EXAM # ______

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 30, 2017, 9:00 a.m. – 12:00 noon

**Work any 5 of the following 6 problems.**

1. Let $X$ be a NLS. Suppose $x \in X$, $\{x_n\}_{n=0}^{\infty}$ is a sequence in $X$, and $M \subset X'$ is such that its span is dense in $X'$. Prove that $x_n \rightharpoonup x$ in $X$ if and only if

   (i) the sequence $\{\|x_n\|\}_{n=0}^{\infty}$ is bounded, and

   (ii) for every $f \in M \subset X'$, $f(x_n) \to f(x)$.

2. Up to a constant multiple, the Legendre polynomial of degree $n$ is

$$P_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n.$$

The Weierstrauss approximation theorem says that for any function $g \in C^0([-1,1])$ and $\epsilon > 0$, there is a polynomial $p$ such that $|g(x) - p(x)| \leq \epsilon$ for any $x \in [-1,1]$.

(a) Show that $P_n$ has exact degree $n$.

(b) Show that the Legendre polynomials form an orthogonal base for $L^2((-1,1))$. [Hint: For orthogonality, show that $P_n$ is orthogonal to $x^m$ for $m < n$ using integration by parts.]

3. Let $X$ be a Banach space and consider $GL(X,X)$, the set of all isomorphisms from $X$ to $X$. Show that $GL(X,X)$ is an open set of $B(X,X)$. $\left[\text{Hint: Recall that } (1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n.\right]$

4. Consider the boundary value problem:

$$\begin{aligned} -u_{xx} + (1+y)u &= f, && \text{for } (x,y) \in (0,1)^2, \\ u(0,y) = 0, \quad u(1,y) &= \cos(y), && \text{for } y \in (0,1). \end{aligned}$$

(a) Find the associated variational problem. In which space should $f$ lie ?

(b) Show that there exists a unique solution to this problem.

5. Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction with contraction constant $\theta \in [0, 1)$ and fixed point $x \in X$. Suppose that $S : X \to X$ is an approximation to $T$ in the sense that for some $\epsilon > 0$,

$$d(T(z), S(z)) \leq \epsilon \quad \text{for all } z \in X.$$

For fixed $x_0 = y_0 \in X$ and integer $m \geq 1$, let $x_m = T(x_{m-1})$ and $y_m = S(y_{m-1})$.

(a) Use induction to show that

$$d(x_m, y_m) \leq \epsilon \frac{1 - \theta^m}{1 - \theta}.$$

(b) We know that $d(x_m, x) \leq \dfrac{\theta^m}{1 - \theta} d(x_0, x_1)$. Use this fact to prove that

$$d(y_m, x) \leq \frac{1}{1 - \theta}\big(\epsilon + \theta^m d(y_0, y_1)\big).$$

6. Fix $g \in L^2(\mathbb{R}^d)$. For any $u \in H^1(\mathbb{R}^d)$, we define

$$J(u) = \int_{\mathbb{R}^d} \left(|\nabla u|^2 + |u|^2 - gu\right)\, dx.$$

(a) Find the Euler-Lagrange equation associated to $J$.

(b) Find all the critical points of $J$ [Hint: You may use the Fourier transform.]

(c) Are those critical points maxima or minima of $J$ ?

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Solutions

1. $X$ NLS, $x \in X$, $\{x_n\} \subseteq X$, $M \subseteq X'$, $\overline{\text{span}(M)} = X'$

$x_n \rightharpoonup x \iff$ (i) $\|x_n\|$ bnded
(ii) $f(x_n) \to f(x)$ $\forall f \in M$.

($\Rightarrow$) If $x_n \rightharpoonup x$, we know that
$f(x_n) \to f(x)$ $\forall f \in X'$, so (ii) holds.

Now for a fixed $f \in X'$,
$|f(x_n)|$ is bounded (since $f(x_n)$ converges)
So $|f(x_n)| = |E_{x_n}(f)| \leq C_f$ $\forall f \in X'$
By UBP,
$$|E_{x_n}(f)| \leq C$$
That is, $\|E_{x_n}\| = \|x_n\|$ bounded.

($\Leftarrow$) Let $g \in X'$, $\varepsilon > 0$ and choose $n$, $\alpha_i \in \mathbb{F}$, $f_i \in M$ for $i = 1, 2, \ldots, n$ s.t.
$$\|g - \underbrace{\sum_{i=1}^{n} \alpha_i f_i}_{h}\| \leq \varepsilon$$
Then
$$g(x_n) - g(x) = g(x_n - x)$$
$$= (g - h)(x_n - x) + h(x_n - x)$$
$\Rightarrow$
$$|g(x_n - x)| \leq \|g - h\| (\|x_n\| + \|x\|) + |h(x_n - x)|$$
$$\leq \varepsilon (M + \|x\|) + |h(x_n - x)|$$
$$\to 0 \quad \text{as} \quad \varepsilon \to 0, \; n \to \infty. \quad //.$$

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2. $P_n = \frac{d^n}{dx^n}(x^2-1)^n$

$g \in C^0, \varepsilon > 0 \Rightarrow \exists\, p$ st. $|g(x)-p(x)| \leq \varepsilon \ \forall x \in [-1,1]$

(a) $(x^2-1)^n \in \mathbb{P}^{2n} \Rightarrow P_n \in \mathbb{P}^n$

Leading term of $(x^2-1)^n$ is $x^{2n}$

$\Rightarrow$

leading term of $P_n$ is $\frac{(2n)!}{n!} x^n$

(b) The set is clearly lin. indep.

For $\perp$, ETS $\perp$ of $P_n$ to $x^m$, $m<n$.

$$\int_{-1}^{1} P_n x^m = \int_{-1}^{1} D^n (x^2-1)^n x^m$$

$$= \underbrace{D^{n-1}(x^2-1)^n x^m \Big|_{-1}^{1}}_{\substack{\text{all terms have} \\ (x^2-1) \Rightarrow \text{term} \\ \text{vanishes}}} - m\int_{-1}^{1} D^{n-1}(x^2-1)^n x^{m-1}$$

$$= \cdots$$

$$= \pm \int_{-1}^{1} D^{n-m+1}(x^2-1)^n \cdot 0 = 0.$$

For density, note for $f \in L^2$, $\exists\, g \in L^2$ s.t. $\|f-g\| \leq \varepsilon$. Weierstrauss gives $p \approx g$.

Now $p \in \text{span}\{P_1, \ldots, P_n\}$ for ~~some~~ $n < \infty$, so

$$\|f-p\| \leq \|f-g\| + \|g-p\|$$

$$\leq \varepsilon + 2\varepsilon = 3\varepsilon \to 0.$$

Thus we have an $\perp$ base.

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3. $X$ Banach. $GL(X,X) \subseteq B(X,X)$

Let $A \in GL(X,X)$

For $\varepsilon$ to be determined, consider

$$B_\varepsilon(A) = \{ T \in B(X,X) : \|T-A\| < \varepsilon \}$$

Now

$$T = T - A + A$$
$$= A(I + A^{-1}(T-A))$$

This is the composition of 2 invertible maps

if (claim) $\|A^{-1}(T-A)\| < 1$

which is true if $\|T-A\| < \frac{1}{\|A^{-1}\|} \equiv \varepsilon.$

To prove the claim [i.e., $I+R$ inv. if $\|R\|<1$ ]

$$S_N = \sum_{n=0}^{N} (-R)^n = I - R + R^2 - \cdots + (-1)^N R^N$$

$\Rightarrow$

$$S_N(I+R) = I + (-1)^N R^{N+1} = (I+R)S_N$$

Now $\|R^{N+1}\| \le \|R\|^{N+1} \to 0$ as $N \to \infty$

Thus $S_N$ is Cauchy $\Rightarrow$ $S_N \to S \in B(X,X)$

So $S_N \to (I+R)^{-1}$.

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4. $\begin{cases} -u_{xx} + (1+y)u = f & (x,y) \in (0,1)^2 \\ u(0,y) = 0, \quad u(1,y) = \cos y \end{cases}$

(a) Let

$$H = \{ v \in L^2((0,1)^2) : v_x \in L^2((0,1)^2) \}$$
$$H_0 = \{ v \in H : v(0,y) = v(1,y) = 0 \quad \forall' y \}$$

The trace at $x = 0, 1$ exists because for a.e. $y$, $v(\cdot, y) \in H^1(0,1)$.

Find $u \in H_0 + x \cos y$ s.t.

$$\overset{L^2((0,1)^2)\ IP}{(u_x, v_x)} + ((1+y)u, v) = (f, v) \quad \forall v \in H_0$$

We want $f \in (H_0)'$

(b) Let $a(u,v) = (u_x, v_x) + ((1-y)u, v)$

Note: $H$ is Hilbert with IP

$\langle u, v \rangle = (u_x, v_x) + (u, v)$

Completeness follows from the completeness of $L^2$: $u_n$ Cauchy $\Rightarrow u_n \xrightarrow{L^2} u$, $u_{n,x} \xrightarrow{L^2} v$

But $\langle u_{n,x}, \varphi \rangle = -\langle u_n, \varphi_x \rangle \to -\langle u, \varphi_x \rangle$
$= \langle u_x, \varphi \rangle$

$\Rightarrow v = u_x$. Thus $u_n \xrightarrow{H} u$.

Now $|a(u,v)| \le \|u_x\| \|v_x\| + 2 \|u\| \|v\| \le 3 \|u\|_H \|v\|_H$

and
$$a(u,u) = \|u_x\|^2 + ((1+y)u, u) \ge \|u_x\|^2 + \|u\|^2$$

x Poincaré $\Rightarrow \|u_x(\cdot,y)\| \ge \gamma \|u(\cdot,y)\| \quad \forall' y \Rightarrow \|u_x\| \ge \gamma \|u\|$

x Thus $a(u,u) \ge \frac{1}{2} \min(1, \gamma^2) \|u\|_H^2$

Lax-Milgram $\Rightarrow \exists!$ sol'n.

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Solutions

5. $(X,d)$ $T: X \to X$ contraction, $\theta$, $Tx = x$

$S: X \to X$, $d(T(z), S(z)) \leq \varepsilon$ $\forall z \in X$.

$x_0 = y_0$, $x_m = T(x_{m-1})$, $y_m = S(y_{m-1})$

(a) We have that

$$d(T(x), T(y)) \leq \theta\, d(x,y)$$

Now

① $d(x_0, y_0) = 0 \leq \varepsilon \dfrac{1-\theta^0}{1-\theta} = 0.$

② Suppose

$$d(x_m, y_m) \leq \varepsilon \frac{1-\theta^m}{1-\theta}$$

Consider

$$\begin{aligned} d(x_{m+1}, y_{m+1}) &= d(Tx_m, Sy_m) \\ &\leq d(Tx_m, Ty_m) + d(Ty_m, Sy_m) \\ &\leq \theta\, d(x_m, y_m) + \varepsilon \\ &\leq \varepsilon\left(\theta \frac{1-\theta^m}{1-\theta} + 1\right) = \varepsilon \frac{1-\theta^{m+1}}{1-\theta} \end{aligned}$$

(b) $d(x_m, x) \leq \dfrac{\theta^m}{1-\theta} d(x_0, x_1)$

$$\begin{aligned} d(y_m, x) &\leq d(y_m, x_m) + d(x_m, x) \\ &\leq \varepsilon \frac{1-\theta^m}{1-\theta} + \frac{\theta^m}{1-\theta} d(\overset{y_0}{\overset{\shortparallel}{x_0}}, x_1) \\ &= \frac{1}{1-\theta}\Big[\varepsilon(1-\theta^m) + \theta^m\big(d(y_0, y_1) + \underbrace{d(y_1, x_1)}_{\leq \varepsilon}\big)\Big] \\ &\leq \frac{1}{1-\theta}\left(\varepsilon + \theta^m d(y_0, y_1)\right). \end{aligned}$$

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6. $g \in L^2$, $u \in H^1$, $J(u) = \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2 - gu)\,dx$

(a) $F(u) = |\nabla u|^2 + |u|^2 - gu$

$$\frac{\partial}{\partial u} F - \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial u_{x_j}}\right) = 0$$

$$\Rightarrow \quad 2u - g - 2\sum_j u_{x_j, x_j} = 0$$

$$\Rightarrow \quad \boxed{-\Delta u + u = \tfrac{1}{2} g}$$

(b) $(1+|\xi|^2)\hat{u} = \frac{1}{2}\hat{g}$

$$\Rightarrow \quad \hat{u} = \frac{1}{2}\,\frac{\hat{g}}{1+|\xi|^2} \quad \Rightarrow \quad u = \frac{1}{2}\left(\frac{\hat{g}}{1+|\xi|^2}\right)^{\vee}$$

$$\Rightarrow \quad u = \frac{1}{2}(2\pi)^{-d/2}\left(\frac{1}{1+|\xi|^2}\right)^{\vee} * g$$

(c) Since

$$J(u+\varepsilon v) - J(u) = \int \left(|\nabla(u+\varepsilon v)|^2 + |u+\varepsilon v|^2 - g(u+\varepsilon v)\right) dx - \int (|\nabla u|^2 + |u|^2 - gu)\,dx$$

$$= \varepsilon \underbrace{\int (2\nabla v \cdot \nabla u + 2uv - gv)}_{=0} + \varepsilon^2 \int (|\nabla v|^2 + |v|^2)$$

$\geqslant 0 \quad \Rightarrow \quad$ minima.

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 31, 2018, 9:00 a.m. – 12:00 noon

**Work any 5 of the following 6 problems.**

**1.** The set $\mathcal{X}$ of all sequences $\{x_n\}_{n=1}^{\infty}$ of complex numbers is a vector space. Let $0 < p < 1$ and let $X \subset \mathcal{X}$ be the set of all sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

**(a)** Show that $X$ is a vector space. [Hint: Show that $|x+y|^p \le 2^{p-1}(|x|^p + |y|^p)$.]

**(b)** Show that the map taking $\{x_n\}_{n=1}^{\infty} \in X$ to $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is *not* a norm on $X$.

**(c)** Show that the map $d : X \times X \to \mathbb{R}$ defined by $d(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a metric on $X$.

**2.** Open Mapping Theorem.

**(a)** State the Open Mapping Theorem.

**(b)** Suppose that $\|\cdot\|$ and $\|\cdot\|'$ are two norms on a vector space $X$. Suppose that both $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are complete and there is a constant $C > 0$ such that

$$\|x\| \le C\|x\|' \quad \text{for all } x \in X.$$

From the Open Mapping Theorem, show that the two norms are equivalent.

**3.** Let $\Omega = [a, b]$, $p, q \in (1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in L^q(\Omega)$. For every $u \in L^p(\Omega)$ define a function $Au$ by setting

$$(Au)(t) = \int_a^t v(s)\, u(s)\, ds \quad \text{for all } t \in \Omega.$$

**(a)** Show that $A$ maps $L^p(\Omega)$ into $L^p(\Omega)$ and is continuous.

**(b)** Explain why $A : L^p(\Omega) \to L^p(\Omega)$ is compact.

**4.** Suppose $(X, d_X)$ and $(Y, d_Y)$ are metric spaces, $Y$ is complete, $A \subset X$ is dense, and $T : A \to Y$ is uniformly continuous. Prove that there is a unique extension $\tilde{T} : X \to Y$ which is uniformly continuous.

**5.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $f \in L^2(\Omega)$ , and $\epsilon > 0$. Suppose $u_\epsilon$ satisfies

$$\begin{aligned} -\epsilon\Delta u_\epsilon + u_\epsilon = f \quad &\text{in } \Omega, \\ u_\epsilon = 0 \quad &\text{on } \partial\Omega. \end{aligned}$$

Show $u_\epsilon \to f$ in $L^2(\Omega)$ as $\epsilon \to 0$. [Hint: Bound appropriate norms of $u_\epsilon$ and $\sqrt{\epsilon}u_\epsilon$.]

**6.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary and outer unit normal $\nu$. Let $\boldsymbol{b}$ a constant vector and $f \in L^2(\Omega)$. Consider the fourth order problem

$$\begin{aligned} u + \Delta^2 u + \boldsymbol{b} \cdot \nabla u = f \quad &\text{in } \Omega, \\ u = 0 \quad \text{and} \quad \nabla u \cdot \nu = 0 \quad &\text{on } \partial\Omega. \end{aligned}$$

**(a)** State the Lax-Milgram Theorem for a real Hilbert space.

**(b)** Develop a suitable variational form for the problem. [Be careful to handle the boundary values and define the Hilbert spaces you use.]

**(c)** Give a hypothesis on $|\boldsymbol{b}|$ so that the Lax-Milgram theorem provides a unique solution to your variational problem. [Hint: Gårding's inequality gives a $C_G > 0$ such that $\|v\|_{H^2}^2 \leq C_G\{\|u\|^2 + \|\Delta u\|^2\}$ for all $v \in H_0^2$.]

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Solutions

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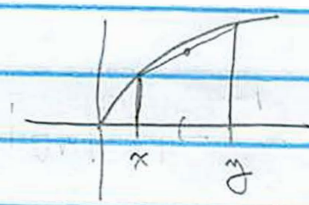
1. $0 < p < 1$ $\quad \{x_n\}$ $\quad \sum |x_n|^p < \infty$

(a) (i) $\{x_n\} + \{y_n\} = \{x_n + y_n\}$



$$f\left(\frac{x+y}{2}\right) < \frac{1}{2}\left(f(x) + f(y)\right)$$

$$\Rightarrow \frac{|x+y|^p}{2^p} < \frac{1}{2}\left(|x|^p + |y|^p\right)$$

$$\Rightarrow \sum |x_n + y_n|^p \le 2^{p-1}\left(\sum x_n^p + \sum y_n^p\right) < \infty$$

(ii) $\alpha\{x_n\} = \{\alpha x_n\}$

$$\sum |\alpha x_n|^p = \alpha^p \sum |x_n|^p < \infty.$$

(b) Consider the $\Delta$ ineq. for

$x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$

$$\Rightarrow \left(\sum |(x+y)_n|^p\right)^{1/p} = 2^{1/p} > 2$$

$$\overset{?}{\le} \left(\sum |x_n|^p\right)^{1/p} + \left(\sum |y_n|^p\right)^{1/p} = 1 + 1 = 2$$

But $2^{1/p} > 2$, so not a norm.

(c) $d(x,y) = \sum |x_n - y_n|^p$

(i) $d(x,y) \ge 0$ ✓, $d(x,y) = 0 \iff x_n = y_n \ \forall n$ ✓.

(ii) $d(x,y) = d(y,x)$ ✓

(iii) $d(x,y) \overset{?}{\le} d(x,z) + d(z,y)$

$$|x_n - y_n|^p = |(x_n - z_n) + (z_n - y_n)|^p$$

$$\le 2^{p-1}\left(|x_n - z_n|^p + |z_n - y_n|^p\right) \quad (\text{see (a)})$$

$$\le |x_n - z_n|^p + |z_n - y_n|^p \quad (p - 1 < 0).$$

$$\Rightarrow \sum |x_n - y_n|^p \le \sum |x_n - z_n|^p + \sum |z_n - y_n|^p. \ ✓$$

2. Open Mapping

(a) Let $X, Y$ be Banach

If $T: X \to Y$ is bounded, linear, surjective

Then $T$ is open (maps open sets to open sets).

(b) $\|\cdot\|, \|\cdot\|'$, $(X, \|\cdot\|)$ & $(X, \|\cdot\|')$ complete.

$\|x\| \le C \|x\|' \quad \forall x \in X.$

Consider

$$i : (X, \|\cdot\|') \to (X, \|\cdot\|) \quad \text{identity}$$

Note $i$ is bounded, surjective (and linear),

so $i$ is open $\Rightarrow$

$$i : B_1' \xrightarrow{\text{onto}} Q \subseteq X \text{ open}$$

Now $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon \subseteq Q$

Given $x \in X$,

$$\frac{\varepsilon}{2} \frac{x}{\|x\|} \in B_\varepsilon \cap B_1'$$

$$\Rightarrow \quad \frac{\varepsilon}{2} \left\| \frac{x}{\|x\|} \right\|' \le 1 \Rightarrow \|x\|' \le \frac{2}{\varepsilon} \|x\|.$$

Thus the norms are equivalent.

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3. $\Omega = [a,b]$, $p, q \in (1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$.
$v \in L^q(\Omega)$. $A : L^p(\Omega) \longrightarrow$ fcns.

$$(Au)(t) = \int_a^t v(s)\,u(s)\,ds \qquad \forall t \in \Omega$$

(a) $A : \longrightarrow L^p(\Omega)$

$$\int_a^b |(Au)(t)|^p = \int_a^b \left| \int_a^t v(s)u(s)\,ds \right|^p dt$$
$$\leq \int_a^b \left( \|v\|_{L^q} \|u\|_{L^p} \right)^p dt$$
$$\leq (b-a)\, \|v\|_{L^q}^p \, \|u\|_{L^p}^p$$

$\Longrightarrow$

$$\|Au\|_{L^p} \leq (b-a)^{1/p} \|v\|_{L^q} \|u\|_{L^p}$$

So $A$ maps into $L^p$ and is continuous.

(b)
$$Au(t) = \int_a^b \underbrace{v(s)\,\chi_{[a,t]}(s)}_{\in L^q(\Omega,\Omega)}\, u(s)\,ds$$

Use density of $C(\Omega)$ in $L^p$, $L^q$

$v_j \xrightarrow{L^q} v$, $u_k \xrightarrow{L^p} u$

$A_j u = \lim_k A_j u_k$, $A_j : C(\Omega) \to C(\Omega)$ is compact by Ascoli-Arzela

$A_j : L^p \to L^p$ compact by density.

$A_j \to A$ is also compact.

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Solutions

4. $X, Y$ metric, $Y$ complete, $A \subseteq X$ dense
$T: A \to Y$ unif. cont. (on $A$).

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0$ s.t. $\forall x, y \in A$,
$d_Y(Tx, Ty) < \varepsilon$ whenever $d_X(x, y) < \delta_\varepsilon$

Let $x \in X$ and $x_n \to x$, $x_n \in A$
Claim: $\{Tx_n\}$ Cauchy.
$\{x_n\}$ Cauchy $\Rightarrow \exists N > 0$ s.t. $d_X(x_n, x_m) < \delta_\varepsilon$
$\forall n, m > N \Rightarrow$
$d_Y(Tx_n, Tx_m) < \varepsilon \quad \forall n, m > N.$
$Y$ complete $\Rightarrow Tx_n \to z \equiv \tilde{T}(x).$

Claim: $\tilde{T}$ unif. cont. If so, then $\tilde{T}x = Tx \; \forall x \in A$
and $\tilde{T}$ is unique (since $A$ dense).
Now $d_Y(\tilde{T}x, \tilde{T}y) \leq d_Y(\tilde{T}x, Tx_n) + d_Y(Tx_n, Ty_m) + d_Y(Ty_m, \tilde{T}y)$
where $x_n \to x$, $y_m \to y$, $x_n, y_m \in A$.
If $N > 0$ chosen so
$d_X(x_n, y_m) < d_X(x_n, x) + d_X(x, y) + d_X(y, y_m)$
$< \delta_\varepsilon/3 + \delta_\varepsilon/3 + \delta_\varepsilon/3$
for $n, m > N$ and $d_X(x, y) < \delta_\varepsilon/3$.
$\Rightarrow d_Y(\tilde{T}x, \tilde{T}y) \leq 3\varepsilon$ for $n, m$ large.

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5. $\Omega \subseteq \mathbb{R}^d$, $f \in L^2$, $\varepsilon > 0$

$-\varepsilon \Delta u_\varepsilon + u_\varepsilon = f$, $\Omega$ ; $u_\varepsilon = 0$, $\partial\Omega$

Equiv. variational form is.

$$\varepsilon(\nabla u_\varepsilon, \nabla v) + (u_\varepsilon, v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

$v = u_\varepsilon \Rightarrow$

$$\varepsilon\|\nabla u_\varepsilon\|^2 + \|u_\varepsilon\|^2 = (f, u_\varepsilon) \leq \|f\|\,\|u_\varepsilon\|$$
$$\leq \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_\varepsilon\|^2$$

$\Rightarrow \quad \varepsilon\|\nabla u_\varepsilon\|^2 + \frac{1}{2}\|u_\varepsilon\|^2 \leq \frac{1}{2}\|f\|^2$

$\Rightarrow \quad \sqrt{\varepsilon}\, u_\varepsilon$ bounded in $H_0^1$

$u_\varepsilon$ " " $L^2$

$\Rightarrow \quad u_\varepsilon \rightharpoonup u$ in $L^2$

$\sqrt{\varepsilon}\, u_\varepsilon \rightharpoonup g$ in $H_0^1 \Rightarrow \sqrt{\varepsilon}\, u_\varepsilon \rightharpoonup g$ in $L^2$

But $\underbrace{\sqrt{\varepsilon}}_{\downarrow 0}\, \underbrace{u_\varepsilon}_{\downarrow u} \rightharpoonup 0 \Rightarrow g = 0$

Thus

$$0 + (u, v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

$\Rightarrow \quad (u - f, v) = 0 \Rightarrow u = f$

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Solutions

6. $\Omega \subseteq \mathbb{R}^d$, $\underline{b}, f \in L^2$

$$\begin{cases} u + \Delta^2 u + b \cdot \nabla u = f, & \Omega \\ u = 0, \quad \nabla u \cdot \nu = 0, & \partial\Omega \end{cases}$$

(a) Let $\mathcal{H}$ be a real Hilbert space with closed subspace $H$. Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be bilinear s.t.

(i) $|B(x,y)| \leq M \|x\| \|y\| \quad \forall x, y \in \mathcal{H}$. (cont.)

(ii) $B(x,x) \geq \gamma \|x\| \quad \forall x \in H$ (coercive)

If $x_0 \in \mathcal{H}$, $F \in H^*$, then $\exists!\, u \in H + x_0$ s.t.

$$B(u,v) = F(v) \quad \forall v \in H.$$

$$\|u\| \leq \frac{1}{\gamma}\|F\| + \left(\frac{M}{\gamma} + 1\right)\|x_0\|$$

(b) Find $u \in H = \{w \in H^2 : w = 0, \nabla w \cdot \nu = 0 \text{ on } \partial\Omega\}$ s.t.

$$(u,v) + (\Delta u, \Delta v) + (b \cdot \nabla u, v) = (f, v) \quad \forall v \in H.$$

(c) $LHS = B(u,v)$, which is cont. For coercivity:

$$\|u\|^2 + \|\Delta u\|^2 + (b \cdot \nabla u, u)$$
$$\geq \|u\| + \|\Delta u\|^2 - |b| \|\nabla u\| \|u\|$$
$$\geq \|u\| + \|\Delta u\|^2 - \frac{4}{\varepsilon}|b|^2 \|\nabla u\|^2 - \varepsilon \|u\|^2$$
$$= (1-\varepsilon)\|u\|^2 + \|\Delta u\|^2 - \frac{4}{\varepsilon}|b|^2 \|\nabla u\|^2$$

Now $C_G(\|u\|^2 + \|\Delta u\|^2) \geq \|\nabla u\|^2 \Rightarrow$ Need

$$(1-\varepsilon)\frac{4}{\varepsilon}|b| < C \quad \Rightarrow \quad \boxed{|b| < \frac{C_G}{4}}$$

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 30, 2019, 9:00 a.m. – 12:00 noon

**Work any 5 of the following 6 problems.**

**1.** Let $X$ be a Banach space with dual space $X^*$ and duality pairing $\langle \cdot, \cdot \rangle$, and let $A, B : X \to X^*$ be linear maps.

**(a)** State the Closed Graph Theorem and what it means for an operator to be closed.

**(b)** Assuming $\langle Ax, y \rangle = \langle Ay, x \rangle$ for all $x, y \in X$, show that $A$ is bounded.

**(c)** Assuming $\langle Bx, x \rangle \geq 0$ for all $x \in X$, show that $B$ is bounded. [Hint: Suppose $B$ is not continuous at 0, so $x_n \to 0$ but $Bx_n \to y \neq 0$. For $w \in X$ such that $\langle y, w \rangle > 0$, consider $x_n + \epsilon w$.]

**2.** Let $\Omega = [0, 1]$ and $1 \leq p < \infty$ be given and consider the sequence of functions $g_n \in L^p(\Omega)$ defined by $g_n(x) = n^{1/p} e^{-nx}$. Show that as $n \to \infty$:

**(a)** $g_n(x)$ converges pointwise to zero for each fixed $x \in (0, 1]$ and for any $p \geq 1$;

**(b)** $g_n$ does not converge strongly to zero in $L^p(\Omega)$ for any $p \geq 1$;

**(c)** $g_n$ converges weakly to zero in $L^p(\Omega)$ if $p > 1$, but not if $p = 1$.

**3.** Prove the Mazur Separation Lemma, which says that if $X$ is a normed linear space, $Y$ a linear subspace of $X$, $w \in X$ but $w \notin Y$, and

$$d = \operatorname{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0,$$

then there exists $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(z) = 0$ for all $z \in Y$. [Hint: Begin by working in $Z = Y + \mathbb{F}w$.]

**4.** Let $\Omega = (0, 1)^2$ and consider the boundary value problem (BVP)

$$-u_{xx} + u_{xy} - u_{yy} = f \quad \text{in } \Omega, \tag{1}$$

$$-u_x + u_y - u = g \quad \text{on } \Gamma_L = \{(0, y) : y \in (0, 1)\}, \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_* = \partial\Omega \setminus \Gamma_L. \tag{3}$$

Let $H = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_*\}$, which is a Hilbert space.

**(a)** Find the corresponding variational problem for $u \in H$ and test functions $v \in H$. Also give the function spaces containing $f$ and $g$.

**(b)** Show the general Poincaré type inequality: There exists $\gamma > 0$ such that

$$\|\nabla v\|^2_{L^2(\Omega)} + \int_{\Gamma_L} v^2 \geq \gamma \|v\|^2_{L^2(\Omega)} \quad \forall v \in H.$$

**(c)** Show that there is a unique solution to the variational problem.

**5.** For fixed $T > 0$, let $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and Lipschitz continuous in the second argument, i.e., there is some $L > 0$ such that

$$\|g(t, v) - g(t, w)\| \leq L \|v - w\| \quad \forall v, w \in \mathbb{R}^d, t \in [0, T],$$

where $\| \cdot \|$ is the norm on $\mathbb{R}^d$. For any $u_0 \in \mathbb{R}^d$, consider the initial value problem (IVP) $u'(t) = g(t, u(t))$ and $u(0) = u_0$.

**(a)** Write this IVP as the fixed point of a functional $G : C^0([0, T]; \mathbb{R}^d) \to C^0([0, T]; \mathbb{R}^d)$.

**(b)** Normally, we use the $L^\infty([0, T])$-norm for $C^0([0, T]; \mathbb{R}^d)$. Show that the function $||| \cdot ||| : C^0([0, T]; \mathbb{R}^d) \to [0, \infty)$, defined by

$$|||v||| = \sup_{0 \leq t \leq T} \left( e^{-Lt} \|v(t)\| \right),$$

is a norm equivalent to the $L^\infty([0, T])$-norm.

**(c)** In terms of this new norm, show that $G$ is a contraction.

**(d)** Explain how we conclude that there is a unique solution $u \in C^1([0, \infty); \mathbb{R}^d)$ to the IVP for all time.

**6.** Consider finding extremals to the problem: Find $u, v \in C^1_{0,1}([0, 1])$ minimizing

$$F(u, v, u', v') = \int_0^1 \left( (u')^2 + (v')^2 + 2uv \right) dx.$$

**(a)** Find the Euler-Lagrange (EL) equations for this problem.

**(b)** Reduce the EL equations to a single equation and find its solution. [Hint: The fourth roots of unity are $\pm 1$ and $\pm i$.]

**(c)** Find the extremal to the problem, up to solving a $4 \times 4$ system of linear equations.

**(d)** If we add the constraint that $\int_0^1 u^2 v' \, dx = 0$, what EL equations do we get?

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Solutions

1. $X$ Banach, $A, B: X \to X^*$ linear.

(a) Closed Graph Theorem:
Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ linear. Then: $T$ is continuous (bnded)
$\iff T$ is closed

$T$ is closed if whenever $x_n \xrightarrow{X} x$, $Tx_n \xrightarrow{Y} y$, then $y = Tx$.

(b) $\langle Ax, y\rangle = \langle Ay, x\rangle \quad \forall x, y \in X$
Suppose $x_n \xrightarrow{X} x$, $Ax_n \xrightarrow{X^*} y$
Then $\langle Ax_n, z\rangle = \langle Az, x_n\rangle \quad \forall z \in X$
$\to \langle y, z\rangle = \langle Az, x\rangle = \langle Ax, z\rangle$
$\Rightarrow Ax = y$, and $A$ continuous (bnded).

(c) $\langle Bx, x\rangle \geq 0 \quad \forall x \in X$
ETS for $x_n \to 0$, $Bx_n \to y \overset{?}{=} 0$
Suppose not: $y \neq 0$ so $\exists\, w \in X$, $\langle y, w\rangle \neq 0$
Consider

$$0 \leq \langle B(x_n + \varepsilon w), x_n + \varepsilon w\rangle$$
$$\to \langle y + \varepsilon Bw, \varepsilon w\rangle$$
$$= \varepsilon\langle y, w\rangle + \underbrace{\varepsilon^2\langle Bw, w\rangle}_{\geq 0}$$

Let $\varepsilon \to 0$ so last term negligible.
Contradiction, since $\varepsilon$ can be $+$ or $-$.
So $y = 0$ and $B$ cont.

2. $\Omega = [0,1]$, $1 \le p < \infty$, $g_n(x) = n^{1/p} e^{-nx}$.

(a) $g_n(x) = \dfrac{n^{1/p}}{e^{nx}} \xrightarrow[\text{L'Hôpital}]{} \dfrac{\frac{1}{p} n^{1/p - 1}}{x e^{nx}} \longrightarrow 0.$

(b) $\|g_n\|_{L^p}^p = \int_0^1 n e^{-npx}\, dx = -\frac{1}{p} e^{-npx}\Big|_0^1$

$= \frac{1}{p}(1 - e^{-np}) \longrightarrow 1/p \ne 0.$

So $g_n \not\to 0$.

(c) Let $h \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

If $p > 1$, then (by density) suppose $h \in C_0^1$.

Then $\exists\ x_* > 0$ s.t. $h(x) \equiv 0$ for $x < x_*$.

Now $\left|\int_0^1 g_n h\right| \le \int_0^1 g_n |h|$,

so suppose $h \ge 0$.

Note $\frac{d}{dn}\left(n^{1/p} e^{-nx} h\right) = n^{1/p} e^{-nx}\left(\frac{1}{pn} - x\right) h$

$\le 0$ for $n$ large enough, and $x \ge x_*$ (so $\forall x$).

Thus $g_n h$ is monotone, so MCT $\Rightarrow$

$\lim \int g_n h = \int \lim g_n h = 0.$

That is, $g_n \rightharpoonup 0$.

But for $p = 1$, $(L^1)^* = L^\infty$. Consider $h \equiv 1$.

Then $\int_0^1 n e^{-nx} = -e^{-nx}\Big|_0^1 = 1 - e^{-n} \longrightarrow 1 \ne 0.$

3. $X$ NLS, $Y$ lin. subsp., $w \in X \setminus Y$.

$$d = \operatorname{dist}(w, Y) = \inf_{y \in Y} \|w - y\| > 0.$$

Work in $Z = Y + \mathbb{F}w$

$z \in Z \Rightarrow \exists!\, y \in Y,\ \lambda \in \mathbb{F}$ s.t.

$$z = y + \lambda w$$

(for otherwise $Y \ni y - y' = (\lambda - \lambda')w \notin Y$).

Let $g : Z \to \mathbb{F}$

$g(z) = \lambda d$ (well defined).

Now $g$ linear and

$$\frac{|g(y+\lambda w)|}{\|y+\lambda w\|} = \frac{|\lambda| d}{\|y+\lambda w\|} = \inf_{z \in Y} \frac{|\lambda| \|w - z\|}{\|y + \lambda w\|}$$

$$= \inf_{z \in Y} \frac{\|-\lambda z + \lambda w\|}{\|y + \lambda w\|} \leq 1.$$

$\Rightarrow \|g\| \leq 1.$

Extend (using Hahn–Banach) to $X$.

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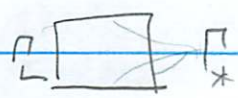
4. $\Omega = (0,1)^2$ $\begin{cases} -u_{xx} + u_{xy} - u_{yy} = f, & \Omega \\ -u_x + u_y + u = g, & \Gamma_L \\ u = 0, & \Gamma_* = \partial\Omega \setminus \Gamma_L \end{cases}$




$H = \{v \in H^1 : v = 0 \text{ on } \Gamma_*\}$

(a) $(u_x, v_x) - \langle u_x, v \rangle_{\Gamma_L}$

$- (u_y, v_x) + \langle u_y, v \rangle_{\Gamma_L}$

$+ (u_y, v_y) \qquad = (f, v)$

$\Rightarrow$

$B(u,v) = (u_x, v_x) - (u_y, v_x) + (u_y, v_y) + \langle u, v \rangle$

$= (f, v) - \langle g, v \rangle_{\Gamma_L}$

So $f \in H^*$, $g \in (H^{1/2}(\Gamma_L))^*$

(b) Suppose not, so $\exists\ v_n$ s.t. $\|v_n\|_{L^2} = 1$

but $\|\nabla v_n\|_{L^2}^2 + \int_{\Gamma_L} v_n^2 \leq \frac{1}{n}$.

$\Rightarrow$ (subseq.)

$\nabla v_n \longrightarrow 0, \quad \int_{\Gamma_L} v_n^2 \longrightarrow 0$

$\|v_n\|_{H^1} \leq 2 \Rightarrow v_n \xrightarrow{H^1} v, \ v_n \xrightarrow{L^2} v$

But $\nabla v = 0 \Rightarrow v = \text{const} \Rightarrow v = 0$,

which contradicts $\|v_n\|_{L^2} = 1$.

(c) Lax-Milgram. Linear form good by $f, g$.

Continuity: $|B(u,v)| \leq (\|u_x\| + \|u_y\|)(\|v_x\| + \|v_y\|) + \|u\|\,\|v\|$

$\leq \|u\|_{H^1} \|v\|_{H^1}$

Coercivity: $B(v,v) \geq \|v_x\|^2 \underbrace{- |(v_y, v_x)|}_{\geq -\frac{1}{2}\|v_y\|^2 + \|v_x\|^2} + \|v_y\|^2 + \int_{\Gamma_L} v^2$

$\geq \frac{1}{2}\gamma \|v\|_{H^1}$ by (b).

5. $u' = g(t, u(t))$, $u(0) = u_0$.

(a) $u(t) - u(0) = \int_0^t g(s, u(s))\,ds$

$\Rightarrow$

$$G(u) = u_0 + \int_0^t g(s, u(s))\,ds$$

(b) $|||v||| = \sup_{0 \le t \le T} \left( e^{-Lt} \|v(t)\| \right)$

Note: $|||v||| \le \|v\|_{L^\infty}$, $|||v||| \ge e^{-LT} \|v\|_{L^\infty}$

so $|||\cdot|||$ equiv. to $\|\cdot\|_{L^\infty}$ $\Rightarrow$
$|||\cdot|||$ satisfies the zero property
Scaling clearly okay

$$|||v+w||| = \sup_t \left( e^{-Lt} \|v+w\| \right) \le \sup_t e^{-Lt} (\|v\| + \|w\|)$$
$$\le \sup_t e^{-Lt} \|v\| + \sup_t e^{-Lt} \|w\|$$
$$= |||v||| + |||w|||$$

so $|||\cdot|||$ is a norm.

(c) $e^{-Lt} \|G(v) - G(w)\| = e^{-Lt} \left\| \int_0^t (g(s,v) - g(s,w)) \right\|$
$$\le L e^{-Lt} \int_0^t \|v-w\| = L e^{-Lt} \int_0^t e^{Ls} e^{-Ls} \|v-w\|$$
$$\le L e^{-Lt} \int_0^t e^{Ls} \; |||v-w|||$$
$$= e^{-Lt} e^{Ls} \Big|_0^t \; |||v-w||| = (1 - e^{-Lt}) |||v-w|||.$$

$\Rightarrow$ $|||G(v) - G(w)||| \le \theta |||v-w|||$, $\theta = 1 - e^{-Lt} < 1$.

(d) Banach contraction mapping Thm $\Rightarrow \exists !$
$u \in C^0([0,T])$ s.t. $G(u) = u$ (i.e., IVP).
But if $u \in C^0$, then $G(u) \in C^1$
$\Rightarrow u \in C^1$.
Finally, let $T \to \infty$.

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Solutions

6. $u, v \in C^1_{0,1}([0,1])$, $F(u,v,u',v') = \int_0^1 \underbrace{[(u')^2 + (v')^2 + 2uv]}_{f} dx$

(a) $f_{y_i} = (f_{y_i'})'$, $i = 1,2$

$$\begin{cases} 2v = 2u'' \\ 2u = 2v'' \end{cases} \Rightarrow \begin{cases} v = u'' \\ u = v'' \end{cases}$$

(b) $u = v'' = u''''$

$\Rightarrow$ $u = e^{rt}$, $r^4 = 1$ $(r = \pm 1, \pm i)$

$u(x) = Ae^x + Be^{-x} + Ce^{ix} + De^{-ix}$

(c) $v(x) = u''$

$= Ae^x + Be^{-x} - Ce^{ix} - De^{-ix}$

$$\begin{cases} u(0) = A + B + C + D = 0 \\ u(1) = Ae + Be^{-1} + Ce^{i} + De^{-i} = 1 \\ v(0) = A + B - C - D = 0 \\ v(1) = Ae + Be^{-1} - Ce^{i} - De^{-i} = 1 \end{cases}$$

(d) $H = \int_0^1 [(u')^2 + (v')^2 + 2uv + \lambda u^2 v'] dx$

$$\begin{cases} 2v + 2\lambda uv' = 2u'' \\ 2u = 2v'' + \lambda(u^2)' \end{cases} \Rightarrow \begin{cases} v + \lambda uv' = u'' \\ u - \frac{1}{2}\lambda(u^2)' = v'' \end{cases}$$

Area A-CAM
Aug 2020
w/ solutions

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

August 7, 2020, about any 3 hours from 9:00 a.m. – 3:00 p.m.

You may use the class textbooks and your own notes on this exam.

**Work any 5 of the following 6 problems.**

**1.** A problem on continuous operators.

**(a)** Define the topological dual of a Banach space.

**(b)** Define the weak topology on a Banach space.

**(c)** Let $X, Y$ be Banach spaces and $A : X \to Y$ be a linear operator. Prove that $A$ is continuous if an only if it is weakly continuous (i.e., it is continuous when $X$ and $Y$ are equipped with their weak topologies).

**Solution.**

**(a)** The topological dual $X'$ of a normed space $X$ consists of all linear and continuous functionals defined on $X$. For a complex space $X$, we may define the topological dual as the space of all *anti*-linear and continuous functionals on $X$. Either space is equipped with the norm

$$l \in X', \quad \|l\|_{X'} := \sup_{x \in X, x \neq 0} \frac{|l(x)|}{\|x\|_X} = \sup_{\|x\|_X \leq 1} |l(x)| = \sup_{\|x\|_X = 1} |l(x)|\,.$$

For a reflexive Banach space, the supremum is actually attained and can be replaced with maximum. The dual space is always complete, no matter whether $X$ is complete or not.

**(b)** The weak topology on a Banach space $X$ is a locally convex topology defined by a family of seminorms

$$X \ni x \mapsto |\langle x', x \rangle| = |x'(x)|, \quad x' \in X'.$$

Due to the definitness of the duality pairing (proved using Hahn-Banach Theorem), the family of seminorms satisfies the axiom of separation which implies that the weak topology is well-defined.

**(c)** We first prove that weak continuity of $A$ implies strong continuity of $A$. Assume, to the contrary, that there exists a sequence $x_n$ such that $\|x_n\|_X \to 0$ but $\|Ax_n\|_Y \not\to 0$. At the cost of replacing $x_n$ with a subsequence, we can assume that there exists $\epsilon > 0$ such that $\|Ax_n\|_Y \geq \epsilon$. Define,

$$\bar{x}_n = \frac{x_n}{\|x_n\|_X^{1/2}}\,.$$

Then,

$$\|\bar{x}_n\|_X = \|x_n\|_X^{1/2} \to 0 \quad \text{and} \quad \|A\bar{x}_n\|_Y \to \infty\,.$$

As the strong convergence implies weak convergence, $\bar{x}_n \rightharpoonup 0$ and, by weak continuity of $A$, $A\bar{x}_n \rightharpoonup 0$ in $Y$. But every weakly convergent sequence must be bounded, a contradiction.

Assume now that $A$ is strongly continuous.

*Lemma:* Let $X$ be an arbitrary topological vector space, and $Y$ be a normed space. Let $A \in \mathcal{L}(X, Y)$. The following conditions are equivalent to each other.

(i) $A : X \to Y$ (with weak topology) is continuous.
(ii) $f \circ A : X \to \mathbb{R}(\mathbb{C})$ is continuous $\forall f \in Y'$.

(i) $\Rightarrow$ (ii). Any linear functional $f \in Y'$ is also continuous on $Y$ with weak topology. Composition of two continuous functions is continuous.

(ii) $\Rightarrow$ (i). Take an arbitrary $B(I_0, \epsilon)$, where $I_0$ is a finite subset of $Y'$. By (ii),

$$\forall g \in I_0 \; \exists B_g, \text{ a neighborhood of } \mathbf{0} \text{ in } X \; : \; x \in B_g \; \Rightarrow \; |g(A(x))| < \epsilon \, .$$

It follows from the definition of filter of neighborhoods that

$$B = \bigcap_{g \in I_0} B_g$$

is also a neighborhood of $\mathbf{0}$. Consequently,

$$x \in B \; \Rightarrow \; |g(A(x))| < \epsilon \; \Rightarrow \; Ax \in B(I_0, \epsilon) \, .$$

To conclude the final result, it is sufficient now to show that, for any $g \in Y'$,

$$g \circ T \; : \; X \; (\text{with weak topology}) \; \to \mathbb{R}$$

is continuous. But $g \circ T$, as a composition of continuous functions, is a strongly continuous linear functional and, consequently, it is continuous in the weak topology as well (compare the discussion in the book).

**2.** Projections on a Hilbert space. Let $X$ and $Y$ be Hilbert spaces, $P : X \to Y$ and $Q : Y \to X$ be bounded linear operators, and suppose that $QP : X \to X$ is an orthogonal projection operator. Let $U_1 = R(QP)$ and $U_2 = N(QP)$, i.e., the image (or range) and null space (or kernel) of the operator, respectively. Moreover, let $V_1 = R(P)$.

**(a)** What does it mean to say $X = U_1 \oplus U_2$? Show that $U_1$ and $U_2$ are orthogonal to each other.

**(b)** Prove that $U_1$ and $V_1$ are isomorphic.

**(c)** Show directly that $P^*Q^* : X \to X$ is an orthogonal projection.

**(d)** If $N(Q) \cap R(PQ) = \{0\}$, show that $PQ : Y \to Y$ is a projection operator (not necessarily orthogonal).

**Solution.**

**(a)** The symbols $X = U_1 \oplus U_2$ mean that $X = \{u_1 + u_2 : u_i \in U_i, i = 1, 2\}$ and $U_1 \cap U_2 = \{0\}$. For $u_i \in U_i$, we know that $u_1 = QPu_1$ and $QPu_2 = 0$, so

$$\langle u_1, u_2 \rangle_X = \langle QPu_1, u_2 - QPu_2 \rangle_X = 0$$

by the definition of orthogonal projection.

**(b)** Consider the map $T = P|_{U_1} : U_1 \to V_1$, that is bounded and linear. Every $v \in V_1$ has some $u \in X$ such that $Pu = v$. However, there are (unique) $u_i \in U_i$ such that $u = u_1 + u_2$, and so $Tu_1 = Pu_1 = Pu = v$ shows that $T$ maps onto $V_1$. To finish, we need to show that $T$ maps one-to-one, i.e., that $Tu_1 = 0$ implies that $u_1 = 0$. But $0 = Tu_1 = Pu_1$, so also $QPu_1 = 0$. Thus $u_1 \in U_1 \cap U_2$, and so $u_1 = 0$.

**(c)** For $u, w \in X$, we compute

$$0 = \langle QPu - u, w \rangle_X = \langle u, P^*Q^*w - w \rangle_X,$$

which shows that $P^*Q^*$ is also an orthogonal projection operator.

**(d)** For $y \in Y$, we know that $QPQPQy = QPQy$, since $QP$ is a projection. But then

$$0 = QPQPQy - QPQy = Q(PQPQy - PQy) = QP(QPQy - Qy)\,.$$

Thus $PQPQy - PQy \in N(Q)$ and clearly $PQPQy - PQy \in R(PQ)$, so $PQPQy = PQy$.

**3.** Hilbert basis. Let $H$ be a separable Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be a maximal orthonormal set (i.e., a Hilbert basis). Let $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and define the linear operator $A : H \to H$ by

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n\,.$$

**(a)** Show that $A$ is continuous and self-adjoint.

**(b)** Show that each $\lambda_n$ is an eigenvalue with eigenvector $e_n$.

**(c)** Show that if $\lambda_n \to 0$, then $A$ is compact. [Hint: Consider the operator $A_N$ defined by a truncated sum, and show that $A_N$ converges to $A$.]

**Solution.**

**(a)** If $x_m \to 0$, then $\|x_m\|^2 = \sum_{n=1}^{\infty} |\langle x_m, e_n \rangle|^2 \to 0$. Thus

$$\|Ax_m\| = \sum_{n=1}^{\infty} |\lambda_n|^2\, |\langle x, e_n \rangle|^2 \le \max_n |\lambda_n|^2 \sum_{n=1}^{\infty} |\langle x_m, e_n \rangle|^2 \to 0\,.$$

That is, $A$ is continuous at 0, and so continuous everywhere.

Now

$$\langle Ax, y \rangle = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \overline{\langle y, e_n \rangle} = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\lambda_n \langle y, e_n \rangle} = \langle x, Ay \rangle$$

is clearly self adjoint (since $\lambda_n$ is real).

**(b)** Compute

$$(A-\lambda I)x = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n - \lambda \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n = \sum_{n=1}^{\infty} (\lambda_n - \lambda)\langle x, e_n\rangle e_n \,,$$

and note that this cannot be invertible when $\lambda = \lambda_n$ for some $n$. Moreover, $Ae_n = \lambda_n e_n$ is clear by orthonormality of the basis.

**(c)** Consider the operators

$$A_N x = \sum_{n=1}^{N} \lambda_n \langle x, e_n\rangle e_n \,.$$

Each has finite dimensional range, and is hence compact. Moreover,

$$\|A_N x - Ax\|^2 = \left\| \sum_{n=N+1}^{\infty} \lambda_n \langle x, e_n\rangle e_n \right\|^2 = \sum_{n=N+1}^{\infty} |\lambda_n|^2 \, |\langle x, e_n\rangle|^2 \to 0 \,,$$

so $A_n \to A$ and $A$ is compact.

**4.** Closed operators. All spaces are real. Consider the operator

$$A \,:\, D(A) \to L^2(0,1), \quad Au = u' + u,$$
$$D(A) := \{u \in L^2(0,1) \,:\, Au \in L^2(0,1),\ u(0) = 0,\ u(1) = 0\},$$

where the derivative is understood in the sense of distributions.

**(a)** Interpret $D(A)$ in terms of Sobolev spaces.

**(b)** Show that $A$ is a closed operator.

**(c)** Prove that $A$ is bounded below in $L^2(0,1)$.

**(d)** Compute the $L^2$-adjoint $A^*$, $L^2(0,1) \supset D(A^*) \ni v \mapsto A^*v \in L^2(0,1)$.

**(e)** Compute the null space of the adjoint operator $A^*$.

**(f)** For an appropriate right-hand side $f$, discuss the well-posedness of the problem:

$$\begin{cases} u \in D(A), \\ Au = f \,. \end{cases}$$

**Solution.**

**(a)** We have

$$u, u' + u \in L^2(0,1) \quad \Leftrightarrow \quad u, u' \in L^2(0,1) \quad \Leftrightarrow \quad u \in H^1(0,1) \,.$$

Consequently, $D(A) = H^1_0(0,1)$.

**(b)** We need to show that

$$D(A) \ni u_n \to u, \quad Au_n \to w \quad \Rightarrow \quad u \in D(A),\, Au = w\,.$$

All convergence is in the $L^2$-sense. Let $\phi \in \mathcal{D}(0,1)$. We have

$$\begin{array}{ccccc} (u_n, -\phi') & + & (u_n, \phi) & = & (-u_n' + u_n, \phi) \to (w, \phi) \\ \downarrow & & \downarrow & & \\ (u, -\phi') & & (u, \phi) & & \end{array}$$

This proves that $-u' + u = w$ and, therefore, $u \in H^1(0,1)$. Moreover, $u_n \to u$ in $H^1(0,1)$. Continuous embedding of $H^1(0,1)$ into $C([0,1])$ implies that,

$$u(x) = \lim_{n\to\infty} u_n(x) = 0 \quad \text{for } x = 0, 1\,.$$

Consequently, $u \in D(A)$.

**(c)** We have

$$\|Au\|^2 = \|u'\|^2 + \|u\|^2 + 2(u', u)\,.$$

But

$$2(u', u) = \int_0^1 \frac{d}{dx}(u^2) = u^2|_0^1 = 0\,.$$

Consequently,

$$\|Au\|^2 = \|u'\|^2 + \|u\|^2 \geq \|u\|^2\,.$$

**(d)** Integration by parts and BC's on $u$ reveal that

$$D(A^*) = H^1(0,1) \quad A^*v = -v' + v\,.$$

**(e)** We get

$$D(A^*) = \{ce^x \,:\, c \in \mathbb{R}\}\,.$$

**(f)** According to the Closed Range Theorem for Closed Operators, the equation has a unique solution $u$ for every $f \in L^2(0,1)$ such that $f \in \mathcal{N}(A^*)^\perp$, i.e.,

$$\int_0^1 f(x)e^x = 0\,.$$

**5.** Variational formulations. Consider the *ultraweak* variational formulation of the previous problem, i.e.,

$$\begin{cases} u \in L^2(0,1) =: U \\ \underbrace{\int_0^1 uA^*v\,dx}_{b(u,v)} = \underbrace{\int_0^1 fv\,dx}_{l(v)} \quad \forall v \in D(A^*) = H^1(0,1) =: V\,, \end{cases}$$

where $A^*$ denotes the $L^2$-adjoint of $A$, $A^*v = -v' + v$, and $f \in L^2(0,1)$. [Hint: For this problem, use results of the previous problem.]

**(a)** Define the operator $B : U \to V'$ and its conjugate corresponding to the bilinear form $b(u,v)$.

**(b)** State the Babuška-Nečas Theorem for Hilbert spaces.

**(c)** Use this theorem to investigate the well-posedness of the variational formulation.

**Solution.**

**(a)** If the bilinear form $b(u,v)$ is continuous (trivially in our case), then the operator

$$B : U \to V', \quad \langle Bu, v\rangle := b(u,v), \quad v \in V,\, u \in U\,,$$

is always well-defined, linear and continuous. The map setting $b$ into $B$ is an isometric isomorphism. The conjugate operator,

$$B' : V'' \sim V \to U', \quad \langle B'v, u\rangle = b(u,v) \quad u \in U, v \in V\,,$$

is also well-defined, linear and continuous with the norm equal to that of $B$.

**(b)** If the bilinear form satisfies the inf-sup condition,

$$\sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U \quad \Leftrightarrow \quad \|Bu\|_{V'} \geq \gamma \|u\|_U$$

and $l \in V'$ vanishes on the null space of the transpose operator,

$$l(v) = 0 \quad \forall v \in V_0 := \{v \in V : b(w,u) = 0 \quad \forall w \in U\}\,,$$

then there exists a unique solution $u$ to the variational problem and

$$\|u\|_U \leq \gamma^{-1} \|l\|_{V'}\,.$$

**(c)** We first prove the inf-sup condition. It is sufficient to find a $v \in H^1(0,1)$ such that $A^*v = u$ and

$$\|v\| \leq C\|A^*v\| = C\|u\|.$$

Once we control the $L^2$-norm of $v$, we control also the $L^2$-norm of its derivative,

$$\|v'\| \leq \|\underbrace{-v' + v}_{A^*v}\| + \|v\| \leq (1+C)\|A^*v\| = (1+C)\|u\|\,,$$

and, consequently,

$$\|v\|^2_{H^1(0,1)} = \|v\|^2 + \|v'\|^2 \leq \underbrace{\left((1+C)^2 + C^2\right)}_{C_1^2} \|u\|^2\,.$$

We have then

$$\sup_v \frac{|b(u,v)|}{\|v\|_{H^1}} \geq \frac{\|u\|^2_{L^2}}{\|v\|_{L^2}} \geq \frac{1}{C_1}\frac{\|u\|^2_{L^2}}{\|u\|_{L^2}} = \frac{1}{C_1}\|u\|_{L^2}\,.$$

Next, we determine the null space of the transpose operator. Clearly,

$$0 = \int_0^1 uA^*v \quad \forall u \in L^2(0,1) \quad \Rightarrow \quad A^*v = 0\,.$$

This gives,

$$\mathcal{N}(B') = \{ce^x \,:\, c \in \mathbb{R}\}\,.$$

Consequently, by the Babuška-Nečas Theorem, for every $l \in (H^1(0,1))'$ that satisfies the compatibility condition

$$l(e^x) = 0\,,$$

the variational problem has a unique solution $u$ that depends continuously upon $l$. Note that the right-hand side may be more general than an $L^2$-function. For the $L^2$-function $f$,

$$l(v) = \int_0^1 fv\,,$$

so the function $f$ must be $L^2$-orthogonal to $e^x$.

Finding a solution $v \in H^1(0,1)$, $A^*v = u \in L^2(0,1)$ is an undetermined problem. We may fix $v$ by adding an extra BC: $v(0) = 0$. You can now find $v$ explicitly (this is an elementary problem), or you can consider an auxiliary problem

$$\begin{cases} v \in H^1(0,1),\ v(0) = 0\,, \\ Lv := -v' + v = u\,. \end{cases}$$

By the same argument as in the previous problem, operator $L$ is bounded below,

$$\| - v' + v\|^2 = \|v'\|^2 + v(1)^2 + \|v\|^2 \geq \|v\|^2\,.$$

The adjoint,

$$D(L^*) := \{u \in H^1(0,1) \,:\, u(1) = 0\}, \qquad L^*u = -u' + u\,,$$

has a trivial null space. The Closed Range Theorem for Closed Operators implies thus that there exists a unique solution $v \in D(L)$, $Lv = A^*v = u$, and $\|v\| \leq \|u\|$.

**6.** Nonlinear equations. Let $X$ be a Banach space and $T : X \to X$ a bounded linear operator. Let $g : X \to X$ be a nonlinear mapping that is $C^1$ and has $g(0) = 0$ and $Dg(0) = 0$. For $f \in X$, we want to solve

$$F(u) = u + Tg(u) = f$$

We consider the map $G(u) = u + \alpha(F(u) - f)$ for some $\alpha \in \mathbb{R}$.

**(a)** Show that $G(u)$ is a contractive map for small enough $u$ and properly chosen $\alpha$.

**(b)** Use the Banach contraction mapping theorem to show that there is a solution to $F(u) = f$, provided $f$ is sufficiently small.

**(c)** Compute $DF(u)(v)$ from the definition of the Fréchet derivative.

**(d)** Solve $F(u) = f$ using the inverse function theorem, provided $f$ is sufficiently small.

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**Solution.**

**(a)** Let $u, v \in X$ and compute

$$G(u) - G(v) = u - v + \alpha(F(u) - F(v)) = (1+\alpha)(u-v) + \alpha T(g(u) - g(v)),$$

so that

$$\|G(u) - G(v)\| \le |1+\alpha|\,\|u-v\| + |\alpha|\,\|T\|\,\|g(u) - g(v)\|.$$

Since $Dg(0) = 0$ and $g$ is $C^1$, given $\epsilon > 0$, there exists $\delta > 0$ such that for $w \in B_\delta(0)$, $\|Dg(w)\| \le \epsilon$. Therefore the mean value theorem shows that

$$\|g(u) - g(v)\| \le \epsilon \|u - v\| \quad \forall u, v \in B_\delta(0).$$

Take, for example, $\alpha = -\frac{1}{2}$ and $\frac{1}{2}\epsilon\,\|T\| < \frac{1}{4}$ (which defines $\delta$). Then $G$ is contractive (with constant $\frac{3}{4}$) on $B_\delta(0)$.

**(b)** It remains to show that $G : B_\delta(0) \to B_\delta(0)$. Compute

$$\|G(u)\| \le \|G(u) - G(0)\| + \|G(0)\| \le \tfrac{3}{4}\|u\| + \|\alpha f\|.$$

Requiring $\|f\| < \dfrac{\delta}{4|\alpha|}$ completes the proof.

**(c)** We compute

$$\begin{aligned} F(u+v) - F(u) &= v + T(g(u+v) - g(u)) = v + T(Dg(u)(v) + R_g(u,v)) \\ &= v + T(Dg(u)(v)) + TR_g(u,v), \end{aligned}$$

where $\|R_g(u,v)\| = o(\|v\|)$. But then $\|TR_g\| \le \|T\|\,\|R_g\| = o(\|v\|)$, so

$$DF(u)(v) = v + TDg(u)(v).$$

**(d)** We note that $F$ is $C^1$ and $DF(0) = I$ is invertible. Thus the inverse function theorem gives open sets $U, V \subset X$ such that $0 \in U$ and $F(0) = 0 \in V$ such that $F$ is a diffeomorphism from $U$ to $V$. Thus we can solve the problem for $f \in V$.

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 28, 2021, about any 3 hours from 9:00 a.m. to 3:00 p.m.

You may use the class textbooks and your own notes on this exam.

**Work any 5 of the following 6 problems.**

**1.** Let the field be real and $\mathbb{P}$ denote the vector space of all polynomials in $x \in \mathbb{R}$; that is, $\mathbb{P} = \left\{ p(x) = \sum_{k=0}^{n} c_k x^k : n \text{ is a nonnegative integer and } c_k \in \mathbb{R} \right\}$. Let $\|\cdot\| : \mathbb{P} \to [0, \infty)$ be defined for such $p$ as $\|p\| = \max_{0 \le k \le n} |c_k|$.

**(a)** Show $\|\cdot\|$ is a norm on $\mathbb{P}$.

**(b)** Show that the NLS $(\mathbb{P}, \|\cdot\|)$ is not complete.

**(c)** Let $m \ge 0$ and $T_m : \mathbb{P} \to \mathbb{R}$ be defined by $T_m p = \sum_{k=0}^{\min(m,n)} c_k$, which is clearly linear. Show that each $T_m$ is bounded.

**(d)** Since $\mathbb{P}$ is not Banach, the Uniform Boundedness Principle need not hold. In fact, show that $\sup_m |T_m p| < \infty$ for each $p \in \mathbb{P}$ but $\sup_m \|T_m\| = \infty$.

**2.** Let $\Omega$ be some set and $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space of functions $f : \Omega \to \mathbb{F}$ ($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Suppose that there is a constant $C(x)$ such that

$$|f(x)| \le C(x)\|f\| \quad \text{for all } f \in H.$$

**(a)** Show that if $f, g \in H$ and $x \in \Omega$, then $|f(x) - g(x)| \le C(x)\|f - g\|$.

**(b)** Show that there exists a function $K : \Omega \times \Omega \to \mathbb{F}$ (called a *reproducing kernel*) such that for each fixed $x \in \Omega$, $K(\cdot, x) \in H$ and

$$f(x) = \langle f, K(\cdot, x) \rangle \quad \text{for all } f \in H.$$

[Hint: Use the Riesz representation theorem.]

**(c)** Show that $K(x, y) = \overline{K(y, x)}$ (i.e., $K$ is conjugate symmetric). Be sure to justify that $K(x, \cdot) \in H$ for each $x \in \Omega$.

**3.** Let $H$ be a complex Hilbert space and $A$ a bounded linear operator on $H$. Define $|A| = (A^* A)^{1/2}$.

**(a)** Show that $|A|$ is a well defined, bounded linear, self-adjoint operator. [Hint: Use Theorem 4.26.]

**(b)** Show that $\|\,|A|x\,\| = \|Ax\|$ for all $x \in H$.

**(c)** Show that $H = \overline{R(|A|)} \oplus N(|A|)$ and that $N(|A|) = N(A)$.

**4.** Half Laplacian in $\mathbb{R}$. Let $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : \ y > 0\}$. For $u \in H^1(\mathbb{R}^2_+)$, we denote by $\bar{u}$ the Fourier transform in $x$ only, i.e.,

$$\bar{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,y) e^{-ix\xi} \, dx.$$

Take $f \in H^1(\mathbb{R})$, and consider $u$ the solution to

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & (x,y) \in \mathbb{R}^2_+, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

**(a)** Find the equation verified by $\bar{u}$.

**(b)** Show that there exists a unique solution of (1) such that $\nabla u \in L^2(\mathbb{R}^2_+)$, and give a formula for $\bar{u}$. [Hint: Solutions to the ODE $y'' - \omega^2 y = 0$ are of the form $Ae^{-\omega t} + Be^{\omega t}$.]

**(c)** For $f \in H^1(\mathbb{R})$, we define $\Delta^\alpha f$, for $0 < \alpha < 1$ a real number, through the Fourier transform as $\widehat{\Delta^\alpha f} = |\xi|^{2\alpha} \hat{f}$. Show that for $u$ solving (1), we have

$$-\partial_y u(x,0) = \Delta^{1/2} f.$$

**(d)** Show that

$$\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx \, dy = \int_{\mathbb{R}} f \Delta^{1/2} f \, dx = \int_{\mathbb{R}} |\Delta^{1/4} f|^2 \, dx.$$

**5.** Let $\Omega \in \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$, and $\alpha > 0$. Consider the boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases}$$

**(a)** For this problem, formulate a variational principle

$$B(u,v) = (f,v) \qquad \forall v \in H^1(\Omega).$$

**(b)** Show that this problem has a unique weak solution.

**6.** Given $I = [0,b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that

$$\begin{cases} u'(s) = g(s) f(u(s)) & \text{for a.e. } s \in I, \\ u(0) = \alpha, \end{cases} \tag{2}$$

where $\alpha \in \mathbb{R}$ is a given constant, $g \in L^p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose that $f$ is Lipschitz continuous and satisfies $f(0) = 0$.

**(a)** Consider the functional

$$F(u) = \alpha + \int_0^s g(\sigma) f(u(\sigma)) \, d\sigma.$$

Show that $F$ maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ is the solution to (2) if and only if it is a fixed point of $F$.

**(b)** Show that there exists $b$ small enough, not depending on $\alpha$, such that $F$ has a unique fixed point in $C^0(I)$.

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1. $\mathbb{P} = \{ p = \sum_{k=0}^{n} c_k x^k \}$

$\|p\| = \max |c_k|$

(a) Norm (i) $\|p\| \geq 0$, $\|p\| = 0 \iff c_k = 0 \ \forall k$
$\iff p = 0$

(ii) $\|cp\| = \| \sum c c_k x^k \| = \max |c c_k|$
$= |c| \max |c_k| = |c| \, \|p\|$

(iii) $\|p+q\| = \max |c_k + d_k|$
$\leq \max |c_k| + \max |d_k| = \|p\| + \|q\|$

(b) Let $p_n = 1 + \frac{1}{2} x + \dots + \frac{1}{n} x^n$

Then $\{p_n\}$ is Cauchy: $(m > n)$

$\|p_n - p_m\| = \frac{1}{n+1} \longrightarrow 0$

But $p_n \not\to p$ with finite degree

(if $\deg p = m$, then $\|p_n - p\| \geq \frac{1}{m+1}$)

Thus $\mathbb{P}$ not complete

(c) $T_m p = \sum_{k=0}^{\min(n,m)} c_k$

$|T_m p| \leq \sum_{k=0}^{\min(n,m)} |c_k| \leq \min(n,m) \|p\|$
$\leq m \|p\|$

(d) $\sup_m |T_m p| \leq n \|p\| < \infty$

$\sup_m \|T_m\| \geq \sup_m \frac{|T_m p|}{\|p\|}$, $p = 1 + x + x^2 + \dots + x^n$

$\geq \sup_m |T_m p| = \min(n,m) = n \to \infty$

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2. $H = \{f : \Omega \to \mathbb{F}\}$

$$|f(x)| \le C(x)\,\|f\| \qquad \forall f \in H$$

(a) $f, g \in H,\ x \in \Omega \implies$

$$|f(x) - g(x)| = |(f-g)(x)| \le C(x)\,\|f-g\|$$

(b) Let $T_x : H \to \mathbb{F}$ be $T_x f = f(x)$

Then $T_x$ is a linear fcnal (by def'n of $+$, sc. mult. of fcns).

Riesz $\implies \exists\ g_x = K(\cdot, x) \in H$ s.t.

$$f(x) = \langle f, K(\cdot, x)\rangle \quad \forall f \in H.$$

(c) $K(\cdot, x) \in H \implies$ (by (b))

$$K(y,x) = \langle K(\cdot, x), K(\cdot, y)\rangle$$

$$= \overline{\langle K(\cdot, y), K(\cdot, x)\rangle} = \overline{K(x,y)}.$$

Note:

$$K(x, \cdot) = \overline{K(\cdot, x)} \in H.$$

3. $H$ complex Hilbert. $A \in B(H,H)$. $|A| = (A^*A)^{1/2}$

(a) Let $T = A^*A \in B(H,H)$

$$\langle Tx, y\rangle = \langle A^*Ax, y\rangle = \langle Ax, Ay\rangle$$
$$= \langle x, A^*Ay\rangle = \langle x, Ty\rangle$$
$$\Rightarrow T = T^*$$

$\langle Tx, x\rangle = \|Ax\|^2 \geq 0 \Rightarrow T \geq 0$

Thm 4.26 $\Rightarrow$ $T$ has a unique pos. sq. root $(A^*A)^{1/2} \in B(H,H)$

Since $(A^*A)^{1/2} \geq 0$, it is self-adjoint.

(b) $\| |A|x \|^2 = \langle |A|x, |A|x\rangle$
$$= \langle |A|^2 x, x\rangle = \langle Tx, x\rangle$$
$$= \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$$

(c) Let $R = \overline{R(|A|)}$

Then $H = R \oplus R^\perp$

$x \in R^\perp \Leftrightarrow \langle x, y\rangle = 0 \quad \forall y \in \overline{R(|A|)}$
$\Leftrightarrow \langle x, y\rangle = 0 \quad \forall y \in R(|A|)$
$\Leftrightarrow \langle x, |A|z\rangle = 0 \quad \forall z \in H$
$\Leftrightarrow \langle |A|x, z\rangle = 0 \quad \forall z \in H$
$\Leftrightarrow x \in N(|A|)$

Thus $R^\perp = N(|A|)$ and
$$H = \overline{R(|A|)} \oplus N(|A|)$$

But
$$x \in N(|A|) \Leftrightarrow \| |A|x \| = 0 \Leftrightarrow \|Ax\| = 0$$
$$\Leftrightarrow x \in N(A)$$

So $N(|A|) = N(A)$.

4. $\mathbb{R}^2_+ = \{(x,y) : y>0\}$. $\bar{u} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x,y)\, e^{-ix\xi}\, dx$

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & \mathbb{R}^2_+ \\ u(x,0) = f(x) \in H^1 \end{cases}$$

(a)
$$\left.\begin{array}{l} \underbrace{\overline{\partial_x^2 u}}_{= -|\xi|^2 \bar{u}} + \partial_y^2 \bar{u} = 0 \end{array}\right\} \Rightarrow \partial_y^2 \bar{u} - |\xi|^2 \bar{u} = 0$$

(b) Note: $\bar{u} = A e^{-|\xi| y} + B e^{+|\xi| y}$

But $\bar{u}(\xi,0) = \hat{f}$ & $\bar{u}$ blows up if $B \neq 0$

so $\bar{u}(\xi,y) = \hat{f}(\xi)\, e^{-|\xi| y}$

$\Rightarrow$

$$u(x,y) = \mathcal{F}^{-1}_\xi\left(\hat{f}(\xi)\, e^{-|\xi| y}\right)$$
$$= (2\pi)^{-1/2} f * \mathcal{F}^{-1}_\xi\left(e^{-|\xi| y}\right)$$

(c) $\widehat{\Delta^\alpha f} = |\xi|^{2\alpha} \hat{f}$

$$-\partial_y u(x,y) = -(2\pi)^{-1/2} f * \left(\partial_y \mathcal{F}^{-1}_\xi\left(e^{-|\xi| y}\right)\right)$$

$$(2\pi)^{-1/2}\left(f * \left(\partial_y \mathcal{F}^{-1}_\xi\left(e^{-|\xi| y}\right)\right)\right)^\wedge = \hat{f}\left(\partial_y \mathcal{F}^{-1}_\xi\left(e^{-|\xi| y}\right)\right)^\wedge$$

$$= -|\xi|\, \hat{f}\left(\mathcal{F}^{-1}_\xi\left(e^{-|\xi| y}\right)\right)^\wedge = -|\xi|\, \hat{f}\, e^{-|\xi| y}$$

as $y \to 0^+$, we have

$$-\partial_y u(x,0) = -\left(-|\xi| \hat{f}\right)^\vee = \Delta^{1/2} f$$

(d)
$$\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx\, dy = (\nabla u, \nabla u)_{\mathbb{R}^2_+} = -(\Delta u, u)_{\mathbb{R}^2_+} + \int_{y=0} \nabla u \cdot \nu\, u$$

$$= \int_{\mathbb{R}} \Delta^{1/2} f\, f$$

$$= \int_{\mathbb{R}} (\Delta^{1/2} f)^\wedge\, \hat{f} = \int_{\mathbb{R}} |\xi|^{1/2} \hat{f}\; |\xi|^{1/2} \hat{f}$$

$$= \int_{\mathbb{R}} |\Delta^{1/4} f|^2\, dx$$

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5. $\alpha > 0$, $f \in L^2$ $\begin{cases} -\Delta u + u = f & , \Omega \\ \partial_\nu u + \alpha u = 0 & , \partial\Omega \end{cases}$

(a) $\underbrace{(-\Delta u, v)}$ $+ (u, v) = (f, v)$

$$= (\nabla u, \nabla v) - \langle \nabla u \cdot \nu, v \rangle$$

$$= (\nabla u, \nabla v) + \langle \alpha u, v \rangle$$

$\Rightarrow$

$$B(u,v) = (\nabla u, \nabla v) + \alpha \langle u, v \rangle$$

$$= (f, v) \qquad \forall v \in H^1$$

where we want $u \in H^1$ as well.

(b) Use Lax-Milgram.

$(f, v)$ gives a cont. lin. fcnal for $f \in L^2 \subseteq (H^1)^*$

$$|B(u,v)| \leq \|\nabla u\| \|\nabla v\| + \alpha \|u\|_{H^1} \|v\|_{H^1}$$

$$\leq M \|u\|_{H^1} \|v\|_{H^1}$$

$$|B(u,u)| = \|\nabla u\|^2 + \alpha \|u\|^2_{L^2(\partial\Omega)} \overset{?}{\geq} \gamma \|u\|_{H^1}$$

We need a Poincaré ineq:

$$\|u\|_{L^2} \overset{?}{\leq} C(\|\nabla u\| + \|u\|_{L^2(\partial\Omega)})$$

Suppose not. Then $\exists\, u_n$ st.

$$1 = \|u_n\|_{L^2} \geq n(\|\nabla u_n\| + \|u_n\|_{L^2(\partial\Omega)})$$

$\Rightarrow$ $u_n \rightharpoonup u \quad H^1 \quad (u_n \to u \quad L^2)$

$\nabla u_n \to 0 \quad L^2$, $\quad u_n|_{\partial\Omega} \to 0 \quad L^2(\partial\Omega)$

$\Rightarrow$ $u = 0$

But $\|u\| = 1$, contradiction.

6. $I=[0,b]$ $\quad \begin{cases} u'(s)=g(s)f(u(s)), & \text{a.e. } s\in I \\ u(0)=\alpha\in\mathbb{R} \end{cases}$

$g\in L^p(I)$, $p\geq 1$; $f:\mathbb{R}\to\mathbb{R}$, $f(0)=0$, Lipschitz

(a)
$$F(u)=\alpha+\int_0^s g(\sigma)\,f(u(\sigma))\,d\sigma$$

$$u\in C^0 \Rightarrow F(u)\in C^0 \Rightarrow F(u)\in L^p$$

$$(F(u))'=g(s)\underbrace{f(u(s))}_{\in L^\infty \text{ since Lipschitz}}\in L^p$$

If
$$u=\alpha+\int_0^s g(\sigma)\,f(u(\sigma))\,d\sigma$$
Then $\begin{cases} u'=g(s)f(u(s)) \\ u(0)=\alpha \end{cases}$

(b) $$\|F(u)-F(v)\|_{L^\infty}=\left\|\int_0^s g(\sigma)\,(f(u)-f(v))\,d\sigma\right\|_{L^\infty}$$
$$\leq \|g\|_{L^p(0,b)}\,\|f(u)-f(v)\|_{L^{\bar{p}}(0,b)}$$
$$\leq \underbrace{\|g\|_{L^p(0,b)}L}_{\theta}\|u-v\|_{L^\infty}$$

$\theta<1$ if $b$ small enough. ($\overset{\text{say}}{\theta=\tfrac{1}{2}}$)

$\Rightarrow F$ contractive

$$\|F(u)\|_{L^\infty}=\|F(u)-F(0)+\alpha\|_{L^\infty}$$
$$\leq \theta\|u\|_{L^\infty}+\alpha\leq \theta R+\alpha \overset{?}{\leq} R$$

$\Rightarrow \alpha\leq(1-\theta)R \Rightarrow R=\dfrac{\alpha}{1-\theta}=2\alpha$

Thus $F:\overline{B_R(0)}\to B_R(0)$

and $\exists!$ fixed pt. in $\overline{B_R(0)}$

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 31, 2022, 9:00 a.m. to 12:00 noon

**Work on any 5 of the following 6 problems.**

**1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\int_\Omega dx = 1$. We consider a real base field and $X \in L^2(\Omega)$ as a *random variable* with *mean* $\mu(X) = \int_\Omega X(x)\,dx$ and *standard deviation* $\sigma(X) = \|X - \mu(X)\|_{L^2(\Omega)}$. The *covariance* of $X, Y \in L^2(\Omega)$ is $\operatorname{cov}(X,Y) = \langle X - \mu(X), Y - \mu(Y)\rangle_{L^2(\Omega)}$.

**(a)** State the domain and range of $\mu$, $\sigma$, and cov. Why is $\mu \in (L^2(\Omega))^*$?

**(b)** Show that $\sigma$ is a seminorm on $L^2(\Omega)$. Why is it *not* a norm?

**(c)** Show that $|\operatorname{cov}(X,Y)| \le \sigma(X)\,\sigma(Y)$.

**(d)** We denote the *probability* that $X \ge \alpha$ as $\operatorname{Prob}(X \ge \alpha) = \int_{\{x : X(x) \ge \alpha\}} dx$. Show Markov's inequality: $\operatorname{Prob}(X \ge \alpha) \le \frac{1}{\alpha}\mu(X)$.

**2.** Let $H$ be a separable, infinite dimensional, complex Hilbert space and $T$ a compact, self-adjoint operator on $H$. The Hilbert-Schmidt and spectral theorems tell us that there is a maximal orthonormal set of eigenvectors $u_n$ with corresponding eigenvalues $\lambda_n$, $n = 1, 2, \ldots$. Let $P_n : H \to H$ be projection onto $\operatorname{span}\{u_n\}$.

**(a)** Show that for all $x \in H$, $P_n x = \langle x, u_n\rangle u_n$, $x = \sum_n P_n x$, and $T = \sum_n \lambda_n P_n$.

**(b)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the property that $f(\lambda) \to 0$ as $\lambda \to 0$. Define $f(T) : H \to H$ by

$$f(T) = \sum_n f(\lambda_n)\, P_n.$$

Show that $f(T)$ is well defined (i.e., the series converges). [Hint: Use Bessel's inequality.]

**(c)** Show that if $f(x) = x^2$, then $f(T) = T^2$.

**3.** Let $T : \mathcal{D}((-1,1)^2) \to \mathcal{D}(-1,1)$ be defined by $(T\varphi)(x) = \varphi(x,0)$.

**(a)** Show that $T$ is a (sequentially) continuous linear operator.

**(b)** Note that the dual operator $T^* : \mathcal{D}'(-1,1) \to \mathcal{D}'((-1,1)^2)$. Determine $T^*(\delta_0)$ and $T^*(\delta_0')$, where $\delta_0$ is the usual Dirac point distribution at 0 in one space dimension.

**4.** Let $\Omega \subset \mathbb{R}^2$ be an open, connected, and bounded domain with a smooth boundary containing 0. Let

$$X = \{f \in W^{1,3}(\Omega) \;:\; f(0) = 0\}.$$

**(a)** Use the Sobolev Embedding Theorem to conclude that $X \subset C^0(\Omega)$ and that $X \ne W^{1,3}(\Omega)$ is a Banach space.

**(b)** Prove the Poincaré-like inequality $\|f\|_{L^3(\Omega)} \le C\|\nabla f\|_{L^3(\Omega)}$, for some constant $C$ independent of $f \in X$.

**5.** Let $f \in L^2(\mathbb{R}^d)$ and consider the problem

$$-\Delta u + u = f \quad \text{in } \mathbb{R}^d.$$

**(a)** Find the variational problem associated to the PDE.

**(b)** Use the Lax Milgram Theorem to show the existence and uniqueness of a solution in $H^1(\mathbb{R}^d)$ to the variational problem.

**(c)** Using the Fourier transform, show that the solution is actually in $H^2(\mathbb{R}^d)$.

**6.** Given $\alpha \in \mathbb{R}$, consider the problem of finding $u$ such that

$$\begin{cases} u'(t) = \dfrac{u(t)}{1 + u^2(t)}, \\ u(0) = \alpha. \end{cases}$$

**(a)** By integrating, rewrite the differential equation in the fixed-point form $u = F(u)$ for an appropriate functional $F$.

**(b)** Show that $F$ maps $C^0([0,T])$ into $C^0([0,T])$ for any $T > 0$.

**(c)** Show that the problem has a unique solution $u \in C^0([0,T])$ for sufficiently small but positive $T$.

1. $\int_\Omega dx = 1$, $X \in L^2(\Omega)$, $\mu(X) = \int X\,dx$, $\sigma(X) = \|X - \mu(X)\|$

$\text{cov}(X,Y) = \langle X-\mu(X), Y-\mu(Y)\rangle$

(a) $\mu : L^2(\Omega) \to \mathbb{R}$, $\sigma : L^2(\Omega) \to \mathbb{R}$, $\text{cov} : L^2(\Omega)\times L^2(\Omega) \to \mathbb{R}$

$\mu(\alpha X + Y) = \int_\Omega (\alpha X + Y) = \alpha \int X + \int Y = \alpha\mu(X) + \mu(Y)$

is linear, and $|\mu(X)| = \int X\,dx \leq \|X\| \cdot \|1\| = \|X\|$

is bounded, so $\mu \in (L^2)^*$.

(b) $\sigma : L^2 \to [0,\infty)$

(i) $\sigma(\alpha X) = \|\alpha X - \mu(\alpha X)\| = \|\alpha(X - \mu(X))\|$

$= |\alpha|\ \|X - \mu(X)\| = |\alpha|\ \sigma(X) \quad \forall\ \alpha \in \mathbb{R}$

(ii) $\sigma(X+Y) = \|X+Y-\mu(X+Y)\| = \|X - \mu(X) + Y - \mu(Y)\|$

$\leq \|X-\mu(X)\| + \|Y - \mu(Y)\| = \sigma(X) + \sigma(Y)$.

Note: if $X = \text{constant} \neq 0$, $\mu(X) = X$, so

$\sigma(X) = 0$ but $X \neq 0$.

(c) $|\text{cov}(X,Y)| = |\langle X - \mu(X), Y - \mu(Y)\rangle|$

$\leq \|X-\mu(X)\|\ \|Y-\mu(Y)\| = \sigma(X)\,\sigma(Y)$

(d) $\text{Prob}(X \geq \alpha) = \int_{\{x : X(x) \geq \alpha\}} dx$

$\leq \int_{\{x : X \geq \alpha\}} \frac{X(x)}{\alpha}\,dx \leq \frac{1}{\alpha}\int X\,dx = \frac{1}{\alpha}\mu(X)$.

since $X \geq 0$

2. $T \in C(H,H)$, $T = T^*$, $\{u_n\}$, $\{\lambda_n\}$, $P_n =$ proj. onto span$\{u_n\}$.

(a) $P_n : H \to \text{span}\{u_n\}$, so for $x \in H$,

$P_n x = c u_n$. But

$$\langle P_n x, u_m \rangle = \langle c u_n, u_m \rangle = c\,\delta_{n,m}$$
$$= \langle x, u_m \rangle \Rightarrow \langle x, u_n \rangle = c$$

$\Rightarrow$
$$P_n x = \langle x, u_n \rangle u_n$$

$x = \sum_m c_m u_m$ (max. ON set)

$$\Rightarrow P_n x = \sum_m c_m P_n u_m = \sum_m c_m \delta_{nm} u_n = c_n u_n$$

$$\Rightarrow x = \sum_n P_n x$$

$$Tx = T\left(\sum_n P_n x\right) = \sum_n T(P_n x) = \sum_n \langle x, u_n \rangle \lambda_n u_n$$

$$\Rightarrow T = \sum_n \lambda_n P_n$$

(b) $f(T) = \sum_n f(\lambda_n) P_n$

Since $\lambda_n \to 0$, $\forall \delta > 0$ $\exists N > 0$ s.t. $|\lambda_n| \le \delta$ $\forall n \ge N$.

Moreover, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(\lambda)| \le \varepsilon$ $\forall |\lambda| \le \delta$.

Thus the tail of the sequence

$$\left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n \right\|^2 = \sup_{\|x\|=1} \left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n x \right\|^2$$

$$\le \sup_{\|x\|=1} \sum_{n=N+1}^{\infty} \varepsilon^2 \|P_n x\|^2 = \sup_{\|x\|=1} \sum_{n=N+1}^{\infty} \varepsilon^2 |\langle x, u_n \rangle|^2$$

$\le \varepsilon^2$ by Bessel's ineq.

(c)
$$T^2 x = T\left(\sum_n \lambda_n P_n x\right) = \sum_m \lambda_m P_m \left(\sum_n \lambda_n P_n x\right)$$

$$= \sum_{m,n} \lambda_m \lambda_n P_m P_n x = \sum_n \lambda_n^2 P_n x$$

$$\Rightarrow T^2 = f(T)$$

3. $T : \mathcal{D}((-1,1)^2) \to \mathcal{D}(-1,1)$, $T\varphi(x) = \varphi(x,y)$

(a) Let $\varphi_n \to \varphi$ in $\mathcal{D}((-1,1)^2)$. Then

$$\| x^{\alpha_1} y^{\alpha_2} D_x^{\beta_1} D_y^{\beta_2} (\varphi_n - \varphi) \|_{L^\infty} \to 0 \quad \forall \alpha_1, \alpha_2, \beta_1, \beta_2$$

Now $\forall\ \alpha, \beta$

$$\| x^\alpha D^\beta (T(\varphi_n) - T(\varphi)) \|_{L^\infty} = \| x^\alpha D^\beta (\varphi_n(\cdot,0) - \varphi(\cdot,0)) \|_{L^\infty}$$
$$\leq \| x^\alpha D_x^\beta \varphi_n(\cdot,\cdot) - \varphi(\cdot,\cdot) \|_{L^\infty((-1,1)^2)} \to 0.$$

$\Rightarrow$ $T$ is seq. cont. Linearity is clear.

(b) For $\mu \in \mathcal{D}'(-1,1)$, $\varphi \in \mathcal{D}((-1,1)^2)$

$$(T^*\mu)(\varphi) = \mu(T\varphi)$$

Thus

$$(T^*\delta)(\varphi) = \delta(T\varphi) = \varphi(0,0)$$

That is, $T^*\delta$ = Dirac dist. at $(0,0)$.

Moreover

$$(T^*\delta')(\varphi) = \delta'(T\varphi) = \delta'(\varphi(x,0))$$
$$= -\frac{\partial \varphi}{\partial x}(0,0)$$

4. $\Omega$ open, connected, bnded, $\ni 0$. $X = \{W^{1,3} : f(0) = 0\}$.

(a) $W_0^{1,3}(\Omega) \hookrightarrow C^0(\Omega)$

since $mp \le d$, i.e. $3 \cdot 1 \ge 2$.

But $\partial\Omega$ smooth, so $W^{1,3}(\Omega) \hookrightarrow C^0(\Omega)$

Thus $X \subseteq C^0(\Omega)$ and $T_0 f = f(0)$

is well-defined cont. op. Thus

$X$ closed under $+$, sc. add $\Longrightarrow X$ Banach, $\neq W^{1}$

(b) Suppose $\|f_n\|_{L^3} = 1$ but $\|\nabla f_n\|_{L^3} \le \frac{1}{n}$.

Then

$\|f_n\|_{W^{1,3}} \le C \Longrightarrow f_{n_i} \rightharpoonup f$ in $W^{1,3}$.

for a subsequence

Moreover $f_{n_i} \longrightarrow f$ in $L^3$

$\nabla f_{n_i} \longrightarrow 0$ in $L^3$

$\Longrightarrow$ $f_{n_i} \longrightarrow f$ in $W^{1,3}$

But $\nabla f_{n_i} \to 0 \Longrightarrow f_{n_i} \longrightarrow$ constant

Since $f_{n_i}(0) = 0$, $f_{n_i} \longrightarrow 0$

This contradicts that $\|f_{n_i}\|_{L^3} = 1$, so

$\|f\|_{L^3} \le C \|\nabla f\|_{L^3}$ for some $C > 0$.

5. $-\Delta u + u = f \in L^2(\mathbb{R}^d)$

(a) Let $v \in \mathcal{D}(\mathbb{R}^d) \underset{\text{dense}}{\subseteq} H^1(\Omega)$. Then

$$(-\Delta u, v) = (\nabla u, \nabla v) \implies$$

$$\begin{cases} \text{Find } u \in H^1(\mathbb{R}^d) \text{ s.t.} \\ (\nabla u, \nabla v) + (u, v) = (f, v) \quad \forall v \in H^1(\mathbb{R}^d) \end{cases}$$

(b) $(f, v) = F(v)$, $F \in (H^1)^*$ since

$$|(f, v)| \le \|f\|_{L^2} \|v\|_{H^1}$$

$$\left| (\nabla u, \nabla v) + (u, v) \right| \le \|\nabla u\| \, \|\nabla v\| + \|u\| \, \|v\|$$
$$\le 2 \|u\|_{H^1} \|v\|_{H^1} \quad \text{continuous.}$$

$$(\nabla u, \nabla u) + (u, u) = \|u\|_{H^1}^2 \quad \text{coercive}$$

$\Rightarrow$ $\exists!$ sol'n in $H^1(\mathbb{R}^d)$.

(c) FT: $\quad |\xi|^2 \hat{u} + \hat{u} = \hat{f}$

$$\Rightarrow \quad \hat{u} \in \frac{\hat{f}}{1 + |\xi|^2}$$

Now $\|u\|_{H^2} = \int (1 + |\xi|^2) |\hat{u}(\xi)|^2 \, d\xi$

$$= \int |\hat{f}(\xi)|^2 \, d\xi = \int |f(x)|^2 \, dx < \infty$$

$\Rightarrow$

$u \in H^2(\mathbb{R}^d)$.

6. $u'(t) = \dfrac{u(t)}{1+u^2(t)}$, $u(0) = \alpha$

(a)
$$u(t) = \alpha + \int_0^t \frac{u(s)}{1+u^2(s)}\,ds \equiv F(u)$$

(b) $F: C^0([0,T]) \longrightarrow C^0([0,T])$
since the integral of a cont. fcn. is cont.

(c) We show $F$ is a contraction on
$X = \{u \in C^0([0,T]) : \|u\|_{L^\infty} \leq R\}$ for some $T, R > 0$.

$$\|F(u) - F(v)\|_{L^\infty} = \left\| \int_0^t \left( \frac{u}{1+u^2} - \frac{v}{1+v^2} \right) dt \right\|_{L^\infty}$$
$$= \left\| \int_0^t \frac{u + uv^2 - v - u^2 v}{(1+u^2)(1+v^2)}\,dt \right\|_{L^\infty} = \left\| \int_0^t \frac{1-uv}{(1+u^2)(1+v^2)}(u-v)\,dt \right\|_{L^\infty}$$
$$\leq T(2R+1)\,\|u-v\|_{L^\infty} \equiv \theta\,\|u-v\|_{L^\infty}$$

Moreover
$$\|F(u)\| = \|F(u) - F(0)\| + |\alpha| \leq T(2R+1)\|u\| + |\alpha|$$
$$\leq T(2R+1)R + |\alpha|$$

Want
$$\theta = T(2R+1) < 1, \quad T(2R+1)R + |\alpha| \leq R$$

Take $R = 2|\alpha| + 1 > 0$. Then
$$T < \frac{1}{4|\alpha|+3} \quad \text{and} \quad T \leq \frac{|\alpha|+1}{(4|\alpha|+3)(2|\alpha|+1)}$$

so take the minimum of these 2.
Note $T > 0$.

EXAM #_____

# CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 15, 2023, 9:00 a.m. to 12:00 noon

**Work on any 5 of the following 6 problems.**

**1.** Let $X$ be a normed linear space and $M \subset X$ a linear subspace.

**(a)** State the Hahn-Banach Theorem for normed linear spaces.

**(b)** If $M$ is closed and $x_0 \in X \setminus M$, use the Hahn-Banach Theorem to prove that there is some $f \in X^*$ satisfying $f(x_0) \neq 0$ and $f(x) = 0$ for any $x \in M$.

**(c)** If $M$ is not necessarily closed, prove that for any $x_0 \in X$, $x_0 \in \overline{M}$ if and only if there is no bounded linear functional $f$ on $X$ satisfying $f(x) = 0$ for any $x \in M$ but $f(x_0) \neq 0$.

**2.** Open Mapping Theorem.

**(a)** State the Open Mapping Theorem.

**(b)** Suppose that $\|\cdot\|$ and $\|\cdot\|'$ are two norms on a vector space $X$. Suppose that both $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are complete and there is a constant $C > 0$ such that

$$\|x\| \leq C\|x\|' \quad \text{for all } x \in X.$$

From the Open Mapping Theorem, show that the two norms are equivalent.

**(c)** Use (b) to show that when $X = L^\infty([0,1])$, $(X, \|\cdot\|_{L^1})$ is *not* complete.

**3.** Let $\varphi \in C_0^\infty(\mathbb{R}^d)$.

**(a)** For $\epsilon > 0$, let $\varphi_\epsilon(\mathbf{x}) = \epsilon^{-d}\varphi(\epsilon^{-1}\mathbf{x})$. Show that for $f \in C^0(\mathbb{R}^d)$,

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \varphi_\epsilon(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x} = C f(0)$$

for some constant $C$. Find the constant $C$.

**(b)** Show that for any $u \in \mathcal{D}'(\mathbb{R}^d)$ and any multi-index $\alpha$, $D^\alpha u * \varphi = u * D^\alpha \varphi$.

**4.** Let $\Omega$ be a bounded domain with a smooth boundary and let $\nu$ be the unit normal vector on its boundary. Consider the solution $(u, v)$ of the differential problem

$$\begin{aligned} u + \Delta^2 u + w &= f \quad \text{in } \Omega, \\ -\Delta w - u &= g \quad \text{in } \Omega, \\ u = \nabla u \cdot \nu &= 0 \quad \text{on } \partial\Omega, \\ w &= \gamma \quad \text{on } \partial\Omega. \end{aligned}$$

**(a)** Provide an appropriate weak form for the problem. In what Sobolev spaces should $u$, $w$, $f$, $g$, $\gamma$, and the test functions lie?

**(b)** Prove that there exists a unique solution to the problem.

**5.** Let $\phi(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $K(x) \in L^1(\mathbb{R})$. Use the contraction mapping principle to prove that the initial-value problem

$$\begin{aligned} \partial_t u &= K * u^2, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \phi(x) \end{aligned}$$

has a continuous and bounded solution $u = u(x,t)$, at least up to some time $T < \infty$.

**6.** For any $a \in \mathbb{R}$ and $b \in \mathbb{R}$, let the Rectified Linear Unit (ReLU) function $R_{a,b} : \mathbb{R} \to \mathbb{R}$ be

$$R_{a,b}(x) = \max(ax + b, 0).$$

Define

$$G = \left\{ \sum_{j=1}^{m} \alpha_j R_{a_j,b_j} \ : \ m \in \mathbb{N},\ \alpha_j, a_j, b_j \in \mathbb{R} \right\}.$$

Clearly $G$ consists of piecewise linear functions. In fact, $\varphi \in G$, where

$$\varphi(x) = R_{0,1}(x) - R_{1,0}(x) + R_{1,-1}(x) - R_{-1,0}(x) + R_{-1,-1}(x) = \begin{cases} 0, & |x| \geq 1, \\ 1 - |x| & |x| \leq 1. \end{cases}$$

**(a)** Show that $G$ is invariant to scaling ($x \mapsto \alpha x$) and translation ($x \mapsto x + c$).

**(b)** Show that if $g \in C([0,1])$, then

$$\int_0^1 R_{a,b}(x)\, g(x)\, dx = 0 \quad \forall\, a, b \in \mathbb{R} \qquad \Longrightarrow \qquad g = 0.$$

**(c)** Let $S$ be the set of functions in $G$ restricted to $[0,1]$. Show that $S$ is dense in $L^2(0,1)$. [Hint: use the density of $C([0,1])$ in $L^2(0,1)$ and (b) to show that $S^\perp = \{0\}$.]