The Importance of Mathematics in the development of Science and Technology

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Abstract

Mathematicians often say that the essence of Mathematics lies in the beauty of numbers, figures and relations, and there is truth in that. But the driving force of mathematical innovation in the last centuries has been the desire to understand how Nature works. This aspect often goes unmentioned.

Together with the experimental method, Mathematics forms the conceptual scheme on which modern science is based and which supports technology, with close interactions among them. Upon these bases the industrial society was born almost four centuries ago, and the new information society is built in the present along the same lines.

In this article we give a brief outline of this scientific connection and how it came to work, with a view to the future and a short comment on Mathematics in Spain.

1 Introduction. Essence and role of Mathematics

Mathematics is an autonomous intellectual discipline, one of the clearest exponents of the creative power of the human mind. On the other hand, it plays a fundamental role in modern Science, has a strong influence on it and it has been influenced by it in an essential way. Here are, briefly presented, two conceptions that symbolize different ways of seeing the great edifice that is present-day Mathematics. These options are reflected in the denominations of Pure and Applied Mathematics. But then, are there two different Mathematics? and, if this true, can they healthily co-exist, or do they actually exist separated from each other? In the present article we will see that, today as in the past, both views of Mathematics are faces of the same coin, looking at times so different, at times so similar.

A first dimension of Mathematics is in fact the *pure* aspect, Mathematics as an art in its own right. It is natural for professional mathematicians to tend to see their science from the point of view of the art in itself, with its postulates, conjectures, lemmas and theorems, with its intuitions and its methods of proof, with its time-honored areas: arithmetic, algebra, geometry and analysis, and the new sprouts: statistics, calculus of probabilities, mathematical logic, computation,... and above all, with its perfect logical deductions. Mathematics is an art that expresses beauty in the form of axioms, theorems and logical or numerical relations; it attracts the researcher precisely because of its logical perfection, by being one of the most compelling examples of the human capacity for reasoning and analysis, by imposing order and harmony where formerly we saw only disorder and chaos. This is the dimension which lies closest to the researcher and, as every pure form of art, it has a fascination that explains why professionals devote an enormous and quite exclusive part of their lives to it. Great scholars, from Pythagoras and Plato to Gauss, have even seen in Mathematics a world of order, more perfect that the everyday physical world. In fact, few professional mathematicians have missed to feel that the true Mathematics inhabits somewhere beyond, in an ideal world, waiting to be discovered by the artist. The idealists can go very far in this direction: thus, Carl G. J. Jacobi sustained once that Mathematics exists only "for the honour of the human mind". Hence, the popular conception, at the same time romantic and misleading, of the mathematician as a distracted *savant* with little or no practical mind.

Indeed, Mathematics is more than that: next to the experimental method, it is the basis upon which modern Science has been built and, as a consequence, the modern technological development rests. It permeates today all aspects of contemporary society from engineering to information, business and finance, not forgetting the movement of the social disciplines toward the status of sciences, which amounts, in other words and with the proper nuances, to the use in these disciplines of the mathematical and experimental methods in combination. The practical importance of Mathematics in Science is indisputable, and it is not under discussion to a certain level, since the overwhelming majority of scientists are well aware of the *instrumental value* of some Mathematics. Thus, a quantitatively very important part of the Mathematics that is taught at universities all over the world is devoted to the education of engineers, physicists, chemists, computer scientists, economists and professionals of several other disciplines. However, the "applied" role of Mathematics goes far beyond this description, is more *essential.* In fact:

(i) Mathematics has played a fundamental role in the formulation of modern Science since the very beginning; a scientific theory is a theory that has an adequate mathematical model;

(ii) the Mathematics that can be applied today covers all the fields of the mathematical science and not only some special topics; it concerns Mathematics of all levels of difficulty and not only simple results and arguments;

(iii) the sciences continue to require today new results from ongoing research and present multiple new directions of inquiry to the researchers, but the rhythm of the contemporary society makes the time lapse substantially shorter and the request more urgent;

(iv) the capabilities of scientific computation have made *numerical simulation* an indispensable tool in the design and control of industrial processes.

In this article we will deal with this aspect whereby *Mathematics is the language* in which the pages of Science are written. thanks to it there has been a development of the combination Science-Technology that has changed the life of the citizen of technologically advanced societies in the last four centuries in a more radical way than the neolithic revolution had done in the ninety previous centuries, and the change has been more dramatic in the last decades than in whole centuries before. Indeed, the daily practice of the physical sciences and engineering hides huge amounts of higher mathematics. Moreover, the very concepts on which their theories are based are essentially *mathematical concepts*. In the last decades we have seen the trend towards mathematization reach other disciplines, like Economics, particularly the financial market, branches of Chemistry, Biology and Medicine, and even the social sciences.

In the hands of the scientist, *Mathematics should permit to assimilate the data and to understand the phenomena.* In the hands of the engineer, it is the tool that makes possible to build a numerical or qualitative *model* whose analysis allows to *make decisions and design artifacts in an efficient and reliable way.* This activity is what, lacking a better name, we call Applied Mathematics. It covers the classical areas like Mathematical Physics and Mathematical Methods for Engineering, but it has today broader contours with the advent of scientific computation and numerical simulation. Applied Mathematics is just the **Mathematics of Reality**, i.e., the real world, whatever this sentence means to each individual reader.

Let us point out that there are other complementary visions of Mathematics: its cultural aspect, its importance in teaching and education as a vehicle for rational thought, its importance in understanding the daily world ("the Mathematics for the common man"), its aspect as a challenging intellectual game. It is at the same time the science of the exact and the calculation of the probable. It is the science of abstract and symbolic reasoning, and it is also today synonymous to computational virtuosity, of capacity to effectively process information, such an important quality in the present world. It tells us about the pure scientist who works with a piece of paper, and also about the world of modelization, computation and control of industrial processes. The layman thinks that Mathematics is tied to the quest of infinite precision. In practice, much of the art of contemporary mathematics is based on estimates. All of these aspects are part of the multiple legacy of Mathematics[1].

We turn next our attention toward the past and present of Applied Mathematics. The reader may find it convenient in a first reading to skip the information contained in the footnotes. Besides, a number of famous and important formulas and equations will appear scattered through the text. They are not meant to be studied as part of this text! The purpose is rather to remind the initiated reader of their beauty and relevance, and at the same time to make the point that there is no *royal way* to Mathematics, namely that a real understanding of the topics outlined here implies serious study.

2 Galileo's and Newton's heirs

Two great historical figures fixed the *key role* of Mathematics in the moments in which modern Science was being born. *Galileo formulated it, Newton demonstrated it.* We ought to add that back in History Pythagoras of Samos (569bC-475bC) sustained that *All is number* and found the wonderful connections between Music and Arithmetic, while Archimedes of Syracuse joined Geometry and Mechanics in the IIIrd century b.C (died 212 b.C.). And one century before Galileo, the universal genius of *Leonardo da Vinci guessed the role* of Mathematics in Science. A pleiad of great mathematicians, the heroes of our story, followed them[2]. The mathematicians who are busy with the application of their art walk truly upon the shoulders of giants[3].

Let us proceed in parts: it is true that from the oldest times Mathematics has been related, even motivated, by practical problems: arithmetics originates from the

[1]We have written about these subjects in [33].

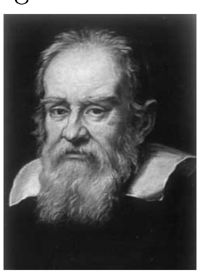
[2]In the story that follows the names of Galileo and Newton are accompanied by other eminent mathematicians, some of which will be assigned a prominent role in the narrative. Such a selection has been useful to set the main hits and to get to know the heroes of our private adventure, but is no doubt unfair from a strictly historical point of view with personalities like Fermat, Leibniz or Gauss, and we want to make it clear at this point. We hope to be excused because of the brevity of the text (the famous *narrow margin* referred to by Fermat) and also because the purpose we have in mind is not the history of science.

[3]Newton's opinion on his predecessors in a letter to R. Hooke, 1675: "If I have seen farther than others, it is by standing on the shoulders of giants". I have endeavoured to include in the text and notes some of the most celebrated phrases of mathematicians and scientists about Mathematics and its application.

activities of counting and adding, geometry stems from measuring lines, surfaces and bodies. But it is also true that Mathematics as a logico-deductive science, just as it was elaborated and bequeathed to us by the Greeks from Pythagoras to Euclides, had a net intellectual, we could say ideal, base that it has always conserved since then and that is a fundamental part of pure Mathematics, that is to say, of Mathematics in itself. This intellectual process lives in its own world and does not owe anything of its merit or beauty to the possible utility or practical application, not more than a poem or a painting do. An easy and frequently made syllogism would lead from here to conclude that the authentic Mathematics lives essentially alien to the adventure of science and technology. We contend that this syllogism is false by a great deal, even if it has been sustained by many mathematicians, and we will make our case clear in what follows by using opinions of famous scientists, but mainly by presenting a record of factual evidence. Indeed, *History shows us that the symbiosis with Science and Technology has been fundamental and fruitful and that Mathematics owes a great deal of its present being and of its main topics to its adventure companions, and conversely the latter to the former.*

As is well known, modern Science appeared in Europe at the end of the Renaissance. It is not based upon Mathematics alone. The fundamental pillar of the building in germ was aptly formulated by the English philosopher and politician Francis Bacon circa 1620 and consists of the *experimental method*[4]. Nature becomes the preferential object of philosophical investigation, we should learn to read and to understand it, and eventually to control it; observation is the means for comprehension and experiment is the test of our predictions. The sciences were formed around this method, first Physics, then Biology, Geology, Chemistry and so on.

Mathematics is, since the very beginning, the other pillar of the sciences. It was Galileo GALILEI (1564-1642) who pointed out in the clearest form that course for the budding sciences at the beginning of the XVII century. His is the famous quotation taken from his letter "Il saggiatore" (The Assayer, 1623) that we reproduce in detail: *"Philosophy is written in that great book that stands constantly open to our gaze, the Universe, but it cannot be understood unless one first learns to comprehend the language in which it is written and its characters. It is written in the language of Mathematics, and its characters are triangles, circles and other geometrical figures,..."*.



GALILEO GALILEI

Galileo was of course a committed defender of the experimental method, to which he contributed his famous astronomical and mechanical observations [5]. The attitude

[4]The inductive method is presented in his work *Novum Organum* or *New Instrument*, 1620.

[5]He wrote down his ideas on Physics, Mathematics and Engineering in the book *Discourses and*

of Galileo had precedents, the most remarkable being as we said Pythagoras and Archimedes in the Ancient Times and Leonardo da Vinci (1452-1519)[6] a century before, but his formulation was determined and put to practice, and it happened in a suitable historical context; it eroded the bases of aristotelism and Scholastics dominant until then in the intellectual world. It bore fruit in a short time and the scientists see themselves reflected in it.

Indeed, philosophies are a small thing if they remain words and polemics, if they are not carried out. The glory of the XVIIth century resides in a series of great philosophers-scientists (called at that time *natural philosophers*), who, without forgetting metaphysics, threw themselves determinedly to the pursuit of the knowledge of Nature and of mathematical invention: René Descartes studied the principles of reasoning, as well as mechanics and the universe; he tied geometry to algebra and wrote "The Discourse of the Method"[7]; Blaise Pascal wrote his "Pensées" but also investigated the principles of fluids (like pressure), geometry, calculus and probabilities. And so did Pierre de Fermat, Edmond Halley, Christiaan Huygens and Gottfried W. Leibniz, a most renowned mathematician and philosopher.

We are ready to meet one of the crucial characters and moments in the history of science. Indeed, the century reaches its culmination with the figure of Isaac NEWTON (1642-1727), who shows the incontestable success of Galileo's proposal as applied to mechanics. He attacks the basic problems debated during the century and
(i) concludes that the movement of solid bodies follows a simple mathematical law that relates the second derivative of space to an invisible *but real* entity, the force. In mathematical words, $\mathbf{F} = m\mathbf{a}$;
(ii) upon applying this theory to the heavenly bodies, he concludes that they move along their orbits in agreement with the law of universal attraction. In formulas, $F = Gmm'/r^2$.


ISAAC NEWTON

mathematical proofs concerning the two new sciences, written in Florence before 1633 but only published abroad in 1638 after the problems with the Church. The two new sciences are mechanics and the science of motion. In 1995 the space probe *Galileo* reached Jupiter and with it the 4 planets discovered by him in 1610.

[6] The interests of Leonardo, a truly universal genius, cover painting and sculpture, engineering and architecture, Physics and Mathematics. Scientist and visionary, he drew the plans of a flying object (forerunner of the helicopter) and coined the term turbulence. Here is a relevant quotation from Leonardo: "No certainty exists where it is not possible to apply the mathematics or in what cannot be related to mathematics".

[7] *Le Discours de la Méthode*, Leiden, 1637, a capital work in the history of science. His work *Les Météores* is considered to be the first attempt to put the study of weather on a scientific basis.

In order to mathematically support the movements resulting from these laws he discovers what we know as infinitesimal calculus and solves differential equations. Moreover, the very formulation of his laws is not possible without the new concepts taken from Differential and Integral Calculus, that carries the names of Newton and Leibniz, and was invented by combining the intuitions of mechanics and geometry. [8]

In 1687, when his monumental work, the *Principia*, is published[9], Mechanics is solidly founded upon the same bases it still has. Mathematics is not only an indispensable tool, *it is the language in which Science is conceived and expressed*, this is the reason of the book's title. From that moment on, the description of the dynamics and evolution of mechanical systems are an essential part of Mathematics. An enormous period of development follows during which Mathematics tries to fulfil this new fundamental role.

Newton is generally considered the most influential scientist in the history of mankind, cf. [29]. Let us provide some additional data in order to better understand the greatness of his legacy. If to his credit we may list the foundations of Mechanics and Astronomy, of Differential and Integral Calculus and Differential Equations, he also studied the nature of light, laid the foundations to Optics and contributed remarkable technical advances, like the refraction telescope. On top of this, he studied the fluids that are today called Newtonian, explained the operation of tides, computed the velocity of sound (and was also interested in Theology, Alchemy and Astrology)[10]. His prestige among his contemporaries was enormous and the most brilliant philosophers of the XVIIIth century (Hume, Kant, Voltaire[11]) studied his work and thought about expanding his fabulous success to all fields of philosophy, a task that turned out to be of a higher difficulty. Indeed, we are still busy with it.

The immensity of the task of understanding Nature did not escape a penetrating person like Newton, with all his success. One of his most celebrated opinions runs as follows: "I do not know what I will look like to others; to myself, I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me".

[8] In placing Newton in proper perspective we have to combine his mathematical formation with the astronomical knowledge he inherited from Tycho Brahe, Johannes Kepler and Galileo.

[9] *Philosophiae Naturalis Principia Mathematica*, London.

[10] He was quite confident in his powers. Here is a quotation from Principia: "From the same principles, I now demonstrate the frame of the System of the World".

[11] It is worth remembering that the Principia were translated into French by the friend of the latter, the Marquise de Châtelet, with his collaboration, 1756. She is described in Encyclopaedia Britannica as "Gabrielle-Émilie Le Tonnelier de Breteuil, Marquise du Ch., French mathematician and physicist who was the mistress of Voltaire", and only in the article her many accomplishments are described.

3 The XVIIIth Century, the century of reason and lights

During the following three centuries, a part of that ocean has been filled with truth, science and Mathematics. Science and Technology, the basis of the Industrial Revolution, have advanced with theories, reasoning and experiments. As a consequence, the society of the XXth century has changed more radically with respect to the XVIIth century than anything that had happened in several thousand years before, since the onset of the great agricultural civilizations. The comfort of house, transportation and communications, and the health of the present-day citizen rest upon technical bases completely unknown to the people of the XVIIth century.

Starting with G.W. Leibniz, a great philosopher and Newton's rival in the famous and a bit sad "dispute of the Calculus", a series of brilliant mathematicians (we would say physicist-mathematicians), like the Bernoulli family, Euler, D'Alembert,... exploited the potential of the new Calculus and formulated mathematically all types of mechanical problems: shooting problems, problems concerning the fall of bodies, the motion of fluids, mechanical vibrations, minimization,...

Infinitesimal methods are likewise powerful in their application to geometry, a discipline that lives in close symbiosis with mechanics. Scholars study the Calculus of Variations, a name for the calculus of minimum values of so-called "functionals", that will bloom in the XXth century as a fundamental topic of Functional Analysis, by then not even foreseen. Jean Le Rond D'Alembert[12] studied the vibration of a string and wrote the wave equation, that led him to decompose a function into a sum of elementary waves, a task also undertaken by Leonhard Euler (1707-1783) who carried out the decomposition into a possibly infinite sum of sinusoidal functions.

Leonhard Euler

Euler is perhaps the most prolific mathematician in history, he made fundamental contributions to Geometry, Analysis and Number Theory, but also to the different branches of Mechanics, Elasticity, Hydrodynamics, Acoustics, and even Music. His Latin is not difficult and his textbooks can be read today with profit and pleasure (preferably after translation!). He lived a great part of his life in St Peterburg, so he is credited with the foundation of Russian Mathematics, together with Daniel Bernoulli.

The problem of infinite sums will worry mathematicians in the near future, but not in these moments of discovery and euphoria, and even less L. Euler whose intuition

[12]a well-known representative of the French Illustration, who combined its brilliant mathematical career with the edition of the famous Encyclopedia, together with Diderot

seems to know no limits.

Some of the glories and griefs of Mathematics as the language of Mechanics can be observed in the study of fluids. A systematic theory escaped even the genius of Newton. Indeed, the most difficult aspect of this theory consisted precisely in finding the exact mathematical hypothesis that permit to build a mathematical model, i.e., to mathematize it *just as it really is* [13]. Toward the year 1738 Johann and Daniel Bernoulli establish the theoretical science of Hydrodynamics on the idealized basis of the so-called *perfect fluids.* The study is continued by Euler, who writes the famous equations (1755)

$$\rho\,(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = 0, \qquad \nabla\cdot\mathbf{u} = 0,$$

(in today's notation) whose analytical solution turns out to be intractable at the time[14]. Moreover, D' Alembert exposes the limitations of the idealization implicit in the concept of perfect fluid by showing that a solid obstacle submitted to a "perfect wind" would suffer no net drag and no net lifting force. This happens because theoretical mechanics does not deal with Nature, that escapes in its pure essence our curiosity, but it rather deals with the mathematical model that we are able to form about it. Experimental agreement allows us to confirm that a theory is good as a model of the physical world, but never that it is perfect[15].

In spite of the relative failure with the fluids, a feeling of optimism invades the minds of the best mathematicians - mechanicians at the end of the century, like Joseph Louis Lagrange or Pierre Simon Laplace. The latter publishes his monumental book "Mécanique céleste" (1788). He is also the author of the "Théorie Analytique des Probabilités", 1812, a most important reference in the development of probability theory. Based on his mechanical studies he thought that the universe functions like a clock (determinism) and declared that the most important mathematical problems were already posed and solved, or about to be solved in a short time. Fortunately, History would prove the great man wrong on these issues. Does this bring to our minds recent heated debates about the end of Physics or History?

Pierre S. Laplace

[13] we recall here Newton's saying about his mechanics: *hypotheses non fingo*, I do not invent the hypothesis or axioms.

[14] and they keep some of their mystery today: the existence of classical solutions given smooth initial data in 3 space dimensions is still an open problem.

[15] we will return to this subject when speaking of Einstein.

4 The XIXth century, the great century of Science

The contribution of the XIXth century to Mathematics, both pure and applied, is surprising by its novelty, by its richness and multiplicity of topics, and by its very unexpectedness. Let us begin our review with the Mathematics that came from Physics.

• Electricity and magnetism: From Michael Faraday to J.C. Maxwell, experiments and partial laws cover a road that counts the names of Gauss, Ampère, Biot, Savart, Lenz, ... till we arrive at the system of partial differential equations that relates the electric and magnetic fields (1863), the work of James Clerk Maxwell[16]

James C. Maxwell

Maxwell's equations are one of the major achievements of Mathematics in the 19th century. Thanks to J.C. Maxwell the new branch of science, whose existence was unsuspected a century before, reached the level of mathematical perfection which Newton accorded to Mechanics. As a consequence, the wave equation is the tool that allows us to describe the propagation of electro-magnetic phenomena in the form of waves characterized by three parameters: first, the amplitude; second, the speed that depends on the medium (and is therefore constant in the vacuum); third, the frequency of oscillation, a variable. In short,

$$u_{tt} = c^2 u_{xx} \quad \Rightarrow \quad u = A\cos(kx - \omega t),$$

where $k = \omega/c$ is called the wave number. Do we need this formula to proceed? The answer is yes, since soon afterwards, and as reflection of the generality of the parameter ω in the mathematical model, Heinrich R. Hertz predicts and discovers electro-magnetic waves out of the visible range (radio waves, 1888), and Guglielmo Marconi discovers wireless telegraphy, that is to say, the radio (1895), introducing us to the world of communications, which is the soul of the XXth century. On the other hand, an incompatibility appears with Newton's mechanics, about which we will speak in a moment. Let this be said about the consequences of the mathematical formulation in the evolution of science.

[16]publication in final form in *Treatise on Electricity and Magnetism*, 1873. Maxwell is considered the major theoretical physicist of the XIXth century, Einstein sustained that Maxwell's work represented the most significant revolution in the study of physics since Newton. The theory of wave propagation is one of the classical branches of applied mathematics nowadays in its multiple variants. An excellent mathematician, Maxwell was an advocate of the probabilistic approach to Science, which he applied to the study of gases, and is credited with saying that "the true Logic for this world is the Calculus of Probabilities"

• THE REAL FLUIDS, from Claude Louis Navier to George Gabriel Stokes, from 1821 to 1856, and later. The Navier-Stokes equations describe real fluids and they govern the behavior of atmospheric phenomena (climate, meteorology, hydrology, the future aeronautics). The correct formulation of the equations describing the movement of real fluids took therefore some 180 years, after the attempts by Newton. A brilliant series of mathematicians figure among the modelers, like S. Poisson and J. C. Saint Venant, as well as the medical doctor J.L.M. Poiseuille, who investigated the blood flow. Lord Kelvin and H. Helmholtz set the bases for the mathematical study of vortices and turbulent fluids, already mentioned by Leonardo, but the full mathematical understanding of the latter is *still an open problem.*

In order not to extend our text excessively we will only mention two further physical theories of great mathematical significance:

• THERMODYNAMICS, which studies the exchange of heat, acquires solid mathematical foundations with James Joule, Saadi Carnot, J.R. Mayer, ... It has strong influence on the calculus with partial derivatives and the concept of exact differential. This theory includes the famous Second Law of Thermodynamics (law of entropy growth in the universe), a fundamental law in science. While its mathematical statement is simple, its practical interpretation has deep implications and puzzles generation after generation of scientists[17].

• Finally, let us mention STATISTICAL MECHANICS, associated to the names of L. Boltzmann and W. Gibbs[18], who carved a branch of Mathematical Physics on the basis of the calculus of probabilities, a discipline that had remained very much at the margin of this scientific adventure[19]. Indeed, the mathematical idealization of chance had been elaborated in the fabulous XVIIth century (ca. 1650) by B. Pascal, P. Fermat and C. Huygens to understand games of chance, and advanced later by Buffon, Bernoulli and Laplace among others. Suddenly, the concept of probability acquires a life of its own in Physics when attempting to model the behavior of huge quantities of particles[20]. This is why the need arises: particles obey of course Newton's mechanical law, but given that Avogadro's number[21] is so huge, approx. 6×10^{23}, it is absolutely impossible to follow individual particle trajectories. Statistical mechanics proposes an average behavior with surprising effectiveness: the prediction of the relationship

[17] with unsuspected consequences: entropy is nowadays a central concept in the Information Theory after the work of C. Shannon, 1948.

[18] not to forget Maxwell, cf. the Maxwell-Boltzmann distribution.

[19] Boltzmann's tomb in Vienna has as sole ornament the entropy formula of statistical mechanics $S = k \log W$.

[20] This was not a trivial step. Boltzmann relied on his belief in atoms, a view strongly opposed at the time by famous scientists like E. Mach.

[21] that measures the number of molecules of a gas per unit volume (22.4 l) under normal temperature and pressure conditions.

between temperature and energy and pressure for a perfect gas is immediate!

We change the scene to portray another of our heroes, a "exemplary life", Bernhard RIEMANN (1826-1866), one of those surprising figures whose work contains the best of pure and applied mathematics. The great German mathematician, who died quite young, is well known as a giant of pure mathematics. He bequeathed to us the hypothesis about the zeros of the "Zeta function" (*Riemann's Hypothesis*) whose proof is considered to be the most famous open problem of Mathematics upon entering the XXI century, after the recent solution of Fermat's conjecture. The Riemann hypothesis asserts that all interesting solutions of the equation $\zeta(s) = 0$ lie on a straight line in the complex plane, precisely at $Re(s) = 1/2$. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every integer solution would shed light on many of the mysteries surrounding the distribution of prime numbers.


BERNHARD RIEMANN

Riemann was a scholar with a geometrical mind who thought of complex analysis in terms of conformal transformations and had the vision of general spaces of several dimensions defined in terms of their local geometry[22]. Today we call them *Riemannian geometries* and they are the foundation upon which theoretical physics is built. Now, the same Riemann studied the propagation of compressible gases and arrived at the conclusion that the mathematical model[23], understood in the sense of classical solutions, is contradictory (because it predicts characteristic lines that intersect each other, so that on them the physical variable would take on several values simultaneously). However, he ventured that the theory was correct if *the point of view were radically changed*; as solutions of the differential equations we must admit functions that are not differentiable, not even continuous. Such boldness, so typical of the best Mathematics of the XIXth and XXth centuries, reminds us again of Newton: Riemann was not "inventing" a theory. The theory of *shock waves* is today a fundamental topic in gas dynamics with its application to Aeronautics, and is therefore one of the most active areas of mathematical research in partial differential equations, ... and engineering.

Inner Evolution. But, even after mentioning Riemann, the present vision would be totally inaccurate if it did not take more explicitly into account the internal evolution of Mathematics, that had by then attained a high level of maturity. We will

[22]his famous article *On the hypotheses which lie at the foundations of Geometry*, in German *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, 1854, published in 1868.

[23]a nonlinear system of partial differential equations of hyperbolic type.

comment only briefly on this issue since it is better known by the mathematical public. The following are some of the star topics. Many of them appeared unexpectedly, but they were meant to have a brilliant future. Let us mention non-Euclidean geometries by J.C.F. Gauss[24], J. Bolyai and N.I. Lobachevski, the rigorous foundation of Infinitesimal Calculus by Augustin L. Cauchy, the theory of functions by Karl Weierstrass, mathematical logic by George Boole and followers, set theory by Georg Cantor, where we mention only a relevant name next to each chapter.

There are research fields in which Mathematics clearly takes the relay from Physics in the task of extracting the substance contained in a concept. This happens with the problem of representing a function as a sum of simple functions, solved by Brook Taylor and Colin McLaurin for sums of powers and posed by Daniel Bernoulli (1753) and Leonhard Euler for trigonometric sums as they appear in the wave and heat equations. Thanks to the insistence of Joseph Fourier (1822)[25] mathematicians enlisted in the adventure of giving a sense to general infinite sums of trigonometric functions,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(\omega x) + b_n \sin(\omega x)\}.$$

This is the origin of a major area of the theory of functions, known as Fourier analysis. The task fraught with baffling difficulties and great successes. Thus, when Paul du Bois Raymond constructed (1873) a continuous and periodic real function whose Fourier series does not converge at all points it seemed that something was quite wrong with the mathematics of wave analysis. On close inspection three options lay open to the researcher: (i) modify the notion of function, (ii) modify the definition of convergence, (iii) replace the basis of sine and cosine functions by better-suited candidates. It is to the credit of mathematicians that all three courses have been pursued with amazing sucess. The fundamental theorem about summation of Fourier series is due to Lennart Carleson, 1966[26], and needs *almost everywhere convergence*, L^2 spaces and the impressive analysis machinery developed in the XXth century[27].

SOCIAL CONTEXT. It may be interesting to say some words on the social evolution of Science in the XIX century. This is the century in which the bourgeois, industrial and democratic revolutions take root in Europe, bringing along the extension of scientific

[24] the "Prince of Mathematicians".

[25] article of 1807, memory presented to the Paris Academy of Sciences and published in 1822.

[26] *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), pp. 135–157.

[27] Here are two quotations from Fourier that will help kindle the debate on Pure versus Applied Mathematics: The first is "The differential equations of the propagation of heat express the most general conditions, and reduce the physical questions to problems of pure analysis, and this is the proper object of theory". Now the second one: "The profound study of nature is the most fertile source of mathematical discoveries".

and industry-related studies in universities and in other specialized centers[28]. That development enlarged exponentially the body of professors and researchers. The advances are so impressive that at the end of the century we find again a frank optimism in the mathematical opinion, if we for instance let ourselves be led by the history written by the German geometer Felix Klein[29]. Another characteristic of this period is the deep separation taking place between mathematicians and physicists and engineers, a consequence of the enormous growth of their respective fields of study. Such a separation will have serious consequences on the evolution of Mathematics in the XXth century, and even on the very concept of Mathematics.

5 An agitated turn of the century

In any case, the turn of the century is spectacular in Physics as in Mathematics. Two extraordinary figures appear in the mathematical arena, Henri POINCARÉ (1854-1912) and David HILBERT (1862-1943). They make a deep imprint in the Mathematics of the XXth century. But a great part of the retrospective brilliance is due to the fact that the turn of century was *a time of crisis*, since the evidence of phenomena that did not fit into the "great explanation" at hand kept mounting.

HENRI POINCARÉ

DAVID HILBERT

• The experiment of Michelson-Morley (1887) showed that the speed of light is really constant, as predicted by the wave theory based on Maxwell's equations. The mechanical model of the world of Euclides-Newton sees a first crack.

• The movement of particles suspended in gases reveals a highly irregular movement, the Brownian movement (Robert Brown, 1827). This is a blow for Euclides' ge-

[28]the first Engineering Schools in Spain appeared in 1834.

[29]*Lectures on the development of mathematics in the 19th century.* Here is a significant quote from Klein: "The great mathematicians like Archimedes, Newton or Gauss always united theory and applications in equal measure".

ometry based on points, straight lines and smooth curves (at least piece-wise smooth).

• The surprises of the theory of functions lead to the Theory of Sets (Georg Cantor) that together with Logic (George Boole, Gottlieb Frege, Giuseppe Peano) form the basis in the attempt to provide rigorous foundations to Mathematics once for all. Mathematics proposes to Science the concepts of consistent and complete theory. Disputes and different schools arises: logicism (Alfred N. Whitehead and Bertrand Russell[30]), intuitionism (Luitzen Brouwer), formalism (D. Hilbert). The paradoxes sow a notable chaos in less strong spirits.

• No efficient analytical or computational tools are available to tackle the complexities of the equations governing continuous media, like fluids. Consequently, the practical Mathematics of engineering plunges into a series of approximations and rules that divorce them from the theory.

• Even the classical questions of the general integration of the equations of movement for three or more (heavenly) bodies turns out to be impossible[31]. Big problems, big remedies: H. Poincaré proposes the qualitative methods and opens the doors to algebraic geometry and topology (called then Analysis Situs, 1895). But, at the time he discovers with his theoretical methods the tremendous complexity hidden in the mathematical model (i.e., the dynamical systems). These monsters are called homoclinical orbits and they will infest with *chaos* the whole body of celestial mechanics when Poincaré is finally well understood (this took several decades). In order to measure the stature of our hero the following quotation could be useful: “in his courses at the Faculté des Sciences de Paris since 1881, and later at the Sorbonne since 1886, Poincaré changed subject every years, touching upon Optics, Electricity, Astronomy, the equilibrium of fluids, Thermodynamics, Light and Probability”.

• We end this summary with some optimistic notes. Thus, the theory of integration of functions is crowned in the works of E. Borel and H. Lebesgue. Now Calculus possesses a concept of integral where the process of taking limits is natural. Functional Analysis is born (Hilbert spaces) and the famous Dirichlet Problem has a solution (in a sense seen then as quite unusual). The price to pay is the construction of a sophisticated mathematical theory that students of science and engineering must absorb, or at least learn to live together with, paraphrasing J. von Neumann.

• Main discoveries of a mathematical nature occur in other sciences and will bear fruit in the next century. The Russian scientist Dmitri I. Mendeleev found order in the chaos of chemical elements and proposed the Periodic Table in 1869, the basis of today’s physico-mathematical treatment of Chemistry. On the other hand, the Austrian monk, botanist and plant experimenter Gregor J. Mendel formulated the

[30]their famous book *Principia Mathematica* dates from 1910.

[31]as exposed by H. Poincaré in his book *Méthodes nouvelles of the mécanique céleste,* Paris, 1899.

rational laws of inheritance, thus laying the mathematical foundation of the science of Genetics[32].

6 The XXth century, a century of wonders

At this height, we expect to have impressed upon the reader a feeling of the deep symbiosis of Mathematics with Physics, of their surprising and in many cases unexpected interactions. By this time this symbiosis includes advanced technological applications, a prelude of what the new century will be. The explosion of Mathematics and Science in the XXth century makes it avisable to reduce our text to some of the most important items. One of the first characteristics that stands out is the progressive mathematization of other sciences, which makes them appear as new horizons for Applied Mathematics.

New Mathematics that came from Physics

• THE THEORY OF RELATIVITY. Albert EINSTEIN, the Man of the Century according to *Time magazine* (year 2000), proposed the two versions of relativity in 1905[33] (special relativity) and in 1916 (general relativity). It will be small surprise to the reader if we say that in both cases it is a matter of an in-depth reflection upon the Mathematics that lie at the basis to Physics. Special relativity has as precursors Lorentz, Poincaré and Minkowski, who studied the invariance group that corresponds to the new geometry of space-time. General relativity uses the geometrical concepts that Riemann elaborated more than a century earlier as a pure thought exercise upon "hypotheses which lie at the foundations of Geometry", and that were developed by the Italian differential geometry school of Ricci, Levi-Civita and Bianchi. Relativity was destined to be a great ballgame for differential geometry in the XXth century. We go from Einstein's equations to the Big Bang and to black holes (Oppenheimer and Snyder, 1939; Penrose and Hawking). All can be seen as an exercise of pure mathematics building a model for a branch of Physics. It is befitting however not to forget the other face of Relativity: since the first experimental confirmation by A. Eddington in 1919, an incessant number of experiments have served to confirm (or rather, with Einstein's mod-

ALBERT EINSTEIN

[32] *Versuche über Pflanzenhybriden* (Experiments with Plant Hybrids), published 1886.

[33] 1905 was the *annus mirabilis* for Einstein. In three separate papers he explained the photoelectric effect, Brownian motion and the theory of relativity. It is unlikely that such a feat will be repeated.

esty, not to refute) the theory of Relativity. Indeed, hypotheses are not invented in the real science[34].

Let us pause to take a look at some of the main formulas. In September 1905 Einstein published a short paper in which he proved the fundamental formula $E = m\,c^2$ about the mathematical equivalence of mass and energy, which has become a classic in the popular culture of the XXth century. On the other hand, the transformation laws of Special Relativity that replace the Galilean transformation laws at high relative velocities, known as Lorentz transformation laws, are:

$$x = \gamma x' + \gamma v t', \qquad t = \gamma t' + \frac{v}{c^2}\gamma x',$$

where the constant γ is called the time dilation factor. It depends on the relative velocity v and is given by the expression: $\gamma = 1/\sqrt{1-(v^2/c^2)}$. Consequently, the addition of velocities follows the surprising rule

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}},$$

very much against what we were used to believe (i.e., $u = u' + v$). On the other hand, Einstein's most recognized formula is of course $E = m\,c^2$, the equivalence of mass and energy, which forms with Planck's quantum formula $E = h\,\nu$ the new vision of energy at the beginning of the century. Precisely, quanta are our next subject.

• QUANTUM MECHANICS. The second magical tour[35] takes us from Max Planck's Hypothesis of the Quanta, 1900, to the Schrödinger Equation (Erwin Schr., 1926) passing by Niels Bohr, Louis de Broglie, Max Born, Werner Heisenberg and Paul Dirac. The door to the atomic world is coded in the marvelous equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\,\psi,$$

where $\hbar$ is the reduced Planck constant, $\hbar = h/2\pi$, $i = \sqrt{-1}$, Δ is the Laplacian operator and $V = V(x, y, z, t)$ is the potential. All this may really seem like a piece of Kabbala, and at first the experts discussed heatedly about the meaning to be given

[34]Here is a significant opinion of Einstein on the role of mathematics: "Mathematics deals exclusively with the relation of concepts to each other without consideration of their relation to experience. Physics too deals with mathematical concepts; however, these concepts attain physical content only by the clear determination of their relation to the objects of experience", in *The theory of Relativity*, 1950. Einstein's opinions are all the more interesting since, contrary to other outstanding figures in the history of Physics, like Newton or Maxwell, he was not himself an outstanding mathematician, at least technically. He left however an impressive legacy to Mathematics through his theories.

[35]quotation in homage to "The Magical Mystery Tour", Lennon and McCartney, 1967.

to the variable $\psi(x, y, z, t)$ called "wave function". Such is the power of Mathematics, these great physicists had found a piece of the Mathematical Code of the Universe but did not know how to interpret the cipher. In 1928 the probabilistic interpretation was proposed by Max Born, where $|\psi|^2$ is the probability density of finding a particle at the location (x, y, z) at the instant t, and this is widely accepted, not without resistance, following Einstein in that[36]. Because Quantum Mechanics is a fundamental challenge to the previously admitted way of looking at the world, to traditional determinism and causality. We may say that Determinism is based on the assumption that "the exact knowledge of the present allows the future to be calculated". Is it not that the dream of the exact sciences, and does not Quantum Mechanics subvert that belief? Pondering on the issue, W. Heisenberg found in 1927 the following answer: "not the conclusion [of the deterministic assumption], but the initial hypothesis is false".

Leaving the world of interpretations aside, we must report that this theory, based on the maximum level of mathematical idealization, will be confirmed by a century of experiments. Its magical part has a stellar moment when Paul A.M. Dirac, using the relativist formulation, proposes the existence of a particle, called today positron (1932), because "the equations admit the sign change with respect to the solution describing the electron",...and the positron was duly discovered[37] by experimental physicists shortly afterwards (Anderson and Blacket, 1932-33). Dirac predicted the existence of the antiproton that was confirmed by Segrè in 1955, and also of the magnetic monopole, but this time existence went without confirmation up to the present day. Dirac's predictions are a remarkable example, in no way unique, where mathematical modeling goes ahead of the experimental evidence[38]. Does this remind us of Hertz?

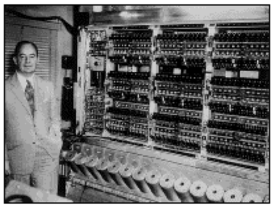
The mathematical harvest is not scarce: the theory of self-adjoint operators in Hilbert spaces with the corresponding spectral theory were developed by John von Neumann (Janos v.N., 1903-1957), one of most versatile geniuses of the century[39],

[36]his famous comment: "God does not play dice".

[37]should we said found? or recognized?

[38]On the other hand, science based solely on mathematical arguments or analogies can be wrong science. Thus, there is strong mathematical tendency to assert that in the realm of particles certain mathematical symmetries are "laws" of nature. A telling counterexample is provided by the law of conservation of parity that specifies that elementary particles and their mirror images *must* behave identically; in 1956-57 three sino-americans T. D. Lee, C. H. Yang and C. S. Wu conjectured and proved that there are subatomic processes that violate that law.

[39]J. von Neumann, *Mathematische Grundlage der Quantenmechanik,* "Mathematical Foundations of Quantum Mechanics", Springer, 1932. Von Neumann's trajectory travels through the most diverse areas of Mathematics, pure and applied: in his youth he modified the ZF set theory, he creates the v.N. algebras in operator theory, he is the father of Game Theory and we will see him later at the Institute for Advanced Studies in Princeton as one of the fathers of the first modern computer. After the war he was busy with hydrodynamics, numerical methods (Monte Carlo, stability for finite



J. V. Neumann

with the purpose of giving sense to the operators that appear in the Schrödinger equation, Laplacians and the rest. He is based on the work of S. Banach and the Italian experts in the Calculus of Variations, but Quantum Mechanics has its whims: it needs some sophisticated mathematical objects, so-called "unbounded linear operators in Hilbert spaces".

We are therefore at the edge or beyond the syllabus of undergraduate Mathematics. This is interesting information for those who claim *that all useful mathematics is necessarily easy*[40]. Together with the Calculus of Variations, Quantum Mechanics has been a continuous source of problems for Functional Analysis, a branch of Mathematics that takes on its own flight.

Mathematics that came from Engineering

• Aeronautics. After the impressive advances of Mathematical Physics in the XIXth century, and in particular of fluid mechanics, it could seem that the old problem of flight, that had already occupied Leonardo da Vinci, had to be solved for good. And the experiments with balloons had been conducted with success a century before[41]. Moreover, the theory of complex variables and of potential and vortex flows had obtained remarkable progress. But with all this progress, real *propelled flight* was not understood nor practiced and a discouraged W. Thomson Lord Kelvin recognized towards the end of the century that the dream of propelled flight was maybe impossible[42]. Then, and after a number of partial successes in different countries, the experimental method is vindicated by the brothers Wilbur and Orville Wright, manufacturers of bicycles and accomplished experimenters with no academic training. They were able to fly a propelled artifact in the inhospitable beaches of Kitty Hawk, North Carolina, in the morning of December 17, 1903. An Engineering discipline is born, Aeronautics. The reaction of the scientific community was immediate and up to the challenge. During the period 1905-10 the main mathematical ingredients missing in the theoretical model were understood (L. Prandtl, M. Kutta, N. E. Zhukovski, S.A. Chaplygin). They deal with the concepts of sustentation, circulation, boundary layer, separation, laminar and turbulent regime. In 30 years the new scientific discipline

difference schemes), the theory of automata, ...

[40] I refer specifically to the opinions of the famous English mathematician G.H. Hardy in his book *A Mathematician's apology*, citeHa, that reflects very different points of view from the ones maintained in this article, cf. specially his section 26. It is a well-known book, of great interest, but time does not seem to have proven the author right. It is to be considered that in 1940 the practical relevance of sophisticated theories like Quantum Mechanics could very well not be clear, as it is today.

[41] Brothers Montgolfier, 1783.

[42] "heavier-than-air flying machines are impossible", he said in 1985.

carries us beyond sound barrier. And with this discipline new branches of applied mathematics see the light, such as the theory of singular perturbations, the theory of supersonic and transonic flows and the mathematical theory of combustion[43].

We restrain here from listing other branches of Engineering that have had a similarly active interaction with Mathematics, but see Section 8.

Great news coming from Mathematics

Mathematics have lived throughout the XX century quite focused on the internal development of the ideas received from the fabulous previous century. Fortunately, the always difficult and generally failed attempt to predict the main lines of the future has an exceptional counterexample in the famous proposal by D. Hilbert at the II International Congress of Mathematicians, celebrated in Paris in 1900. Hilbert summarized in 23 problems the main challenges faced by Mathematics, going from the most theoretical aspects of pure mathematics to the problems of mathematical physics[44], cf. reference [13]. Those 23 problems have been of great importance in the course of the century, but other lines have come to complement them. Let us point out three important developments among many others.

• THE CALCULUS OF PROBABILITIES. It may look like an answer to the needs presented by Quantum Mechanics, but in reality it happened independently. In the 30's Andrei N. Kolmogorov put in Moscow the foundations of axiomatic probability[45] upon set theory, and abstract measure theory is born. The names of P. Levy in France and N. Wiener in the USA are usually associated with this discovery. We should not forget the precedents: Boltzmann studied the Brownian motion and Einstein obtained the Nobel Prize in 1921, not for the theory made him famous, but for his studies on the photo-electrical effect and... on Brownian motion. Markov chains had been studied since 1900 by A.A. Markov. Nowadays, the theory of Stochastic Processes is a main area of this booming branch of Mathematics, and the Itô derivative and integral are tools of continuous stochastical analysis to be compared to the classical infinitesimal calculus of Newton and Leibnitz. All this developement was completely unknown, even unsuspected, to older ages and it takes upon itself the task of informing us about uncertain and random events and their probable outcome or evolution. It is not just an academic pursuit, it has very important applications in scientific, industrial and financial processes.

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[43]More toward theoretical mathematics we have the mathematical theories of front propagation and that of singularity formation, like blow-up for nonlinear differential equations.

[44]Though it must be said that these latter were relatively under-represented, and Hilbert worked on the subject in subsequent years.

[45]His book *Grundbegriffe der Wahrscheinlichkeitsrechnung*, "Foundations of the Calculus of Probabilities", was published in 1933.

• DETERMINISTIC CHAOS. The study of chaos generated by differential equations, already announced by Poincaré, whose Mathematics had matured thanks to the efforts of different mathematicians, especially G. Birkhoff, had to wait for the work of a physicist devoted to the study of weather to acquire a dramatic impulse. In effect, this merit is attributed to Edward Lorenz, from MIT [46]. Interested in the study of convective processes in the atmosphere, he proposed a very simplified model consisting of three ordinary differential equations and I will not resist the temptation of reproducing it for you

$$\begin{cases} x' = -10x + 10y, \\ y' = 28x - y + xz, \\ z' = \frac{8}{3}z + xy. \end{cases}$$

For this particular chioce of parameters he found to his surprise that the numerical trajectories produced by the computer do not converge to a periodic solution. The 12 page paper dates from 1963. Deterministic chaos is born, along with strange attractors and a whole branch of Mathematics, at the beginning quite experimental, then theoretical, a great novelty made possible by the advent of the computer. Authors like S. Smale., D. Ruelle and M. Feigenbaum become world-famous[47]. Objects like the *fractal sets* of B. Mandelbrot[48], already announced in the work of G. Julia in the 1920's, enter the scene. The study of fractal, chaotic and turbulent processes is one of the border-lines of present mathematical thought, the relation of deterministic chaos to natural chaotic and turbulent phenomena still being largely unknown.

• NEW CONCEPTS OF SOLUTION IN DIFFERENTIAL EQUATIONS. Toward the 1930's it was clear for many researchers that the concept of classical solution was not sufficient to build a theory of differential equations for use in mathematical physics which would satisfy the requirements of the applied science. In effect, it is natural in this discipline to work with *problems*, i.e., with sets of equations and additional data, and to require them to be *well posed.* Following J. Hadamard, this means that such problems should have a solution, that this one has be to unique if sufficient data are given, and finally that the solution should depend continuously on the data. Now, it may happen that classical solutions do not to exist and this fact canin the real science even be proved in a rigorous way, and even then the problem could be reasonable from the physical point of view. Or it may simply happen that the concept of solution whose existence turns out to be natural and simple to show is not the classical concept. Faced with this challenge mathematicians have developed a diverse set of notions of *generalized solutions* with physical meaning. A remarkable

[46] *Deterministic non-periodic flow*, J. Atmos. Sci **20** (1963), 130–141.

[47] cf. Ian Stewart, *Does God play dice? The New Mathematics of Chaos*, Penguin, London, 1989.

[48] cf. B. Mandelbrot, *The fractal geometry of Nature,* 2nd ed., San Francisco, 1982.

example arises in Dirichlet's problem of energy minimization already mentioned[49]. Another basic example arises with Riemann's problem of gas dynamics. Yet another similar problem is tackled by J. Leray (1933) in the study of the solutions of the Navier-Stokes equations for real (viscous) fluid in tridimensional space. Thanks to the work of functional analysts (S.L. Sobolev, L. Schwartz, ...) the concepts of *weak solution* and *solution in the sense of distributions* are developed to suit those needs. Summarizing a great deal, the main idea is not to ask the solutions to possess all the derivatives implicit in the equation, but rather to comply with a family of tests. With the experts in conservation laws (P. Lax, O. A.Oleinik, S.N. Kruzhkov) we arrive at the concept of *entropy solutions*, needed for gas dynamics where weak solutions are insufficient. Entropy solutions of gas dynamics equations "solve" the differential equations but may not even be continuous (and thus we recover the legacy of Riemann, Rankine and Hugoniot and their shock waves).

In our days new concepts of solution appear to suit new needs, such as the *viscosity solutions* of M.G. Crandall, L.C. Evans and P.L. Lions. L. Caffarelli extends the concept to the problems of phase transition or free boundary, in which the discontinuity is a fundamental part of the mathematical setting. And the saga continues with so-called mild solutions, semigroup solutions, renormalized solutions,...

One of the most striking aspects of these new concepts is their compatibility with the *numerical solutions* produced by the discrete methods of numerical calculus. We find thereby a surprising alliance of the abstract and the numerical concepts against "the unflexibility of the classical concepts".

7 Engineering and Mathematics in the last revolution of the century. Computers and computational mathematics

The practical realization of the old dream of building a calculating machine takes shape in form of the modern computer that originates from two sources, Technology and Mathematics. Both combine towards a fabulous invention in the year 1946[50]. From one side, we have the old project of the calculating machine, already thought

[49]It deals with minimizing the energy integral $\int_\Omega |\nabla u|^2\,dx$ among all the admissible functions $u = u(x)$ defined in a domain of the space, Ω, and which take assigned values on the border of Ω; ∇u dentoes the gradient of u. The crucial question in order to envisage the correct solution, is to decide what is understood under the label *admissible* function. The answer motivates Hilbert spaces.

[50]with this date I refer to the ENIAC computer.

of by B. Pascal[51] and G. Leibniz in the XVIIth century[52], which owes so much to Ch. Babbage at the beginning of the XIXth century, and finally is to be realized in the XX century in an efficient form thanks to the progress of electronics: first, the vacuum tube and then a line of impressive technical progress that leads to the semiconductor, to miniaturization and the *chip*[53]. But the computer is not born as a passive calculating machine, it is born with a program. This is the legacy of mathematical logic, from G. Boole with his algebra to the program of formalization of Mathematics by D. Hilbert, that leads to Kurt Gödel's incompleteness proof in 1931[54], one of the absolute Mathematical Hits in the XXth century. Which in turn provokes the interest of a mathematical genius, Alan TURING (1912-1954), who translates the program of formalization to the language of machines (*On Computable Numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, 1937), and invents, together with Alonzo Church, Computability theory. All this happened years before a physical computer was to see the light. There follows a historical moment: the war effort, deciphering the German code Enigma,... Enters von Neumann and ENIAC is built in 1946[55]. In the short period of 50 years we have seen the evolution from huge machines, that could handle kilobytes to megabytes, to the personal computer with capacity of several gigas and to the World Wide Web. Duality in the computer world continues in the form of the couple Hardware and Software[56].

A. TURING

THE COMPUTATIONAL WORLD, A NEW WORLD FOR MATHEMATICS. The computer world is changing little by little the daily life of the citizen: banking transactions, electronic mail, ticket reservations,... Its effect upon Mathematics, less known by the general public, is even more dramatic. On the one hand, new branches appear like theoretical Computational Mathematics, or the theory of automata and formal languages. But all branches of Mathematics, pure and applied, are affected by the sudden

[51]his *machine à calculer*, the *Pascaline*, is famous.

[52]Leibiniz thought in the direction of algebra and symbolic logic. Recent investigations indicate that the first of such calculating machine is due to a German, Schickard, 1623

[53]the integrated circuit was invented by R. Noyce and J. Kilby in 1958

[54]the incompleteness of formal systems, was published in *Ueber formal unentscheidbare Saetze der Principia Mathematica und verwandter Systeme*, "On formally undecidible propositions...".

[55]Mention should be made of the English Colossus, 1942, and the German Z1 to Z4 machines, cf. ref. [18]. ENIAC appears as a calculating machine with three characteristics: electronic, digital and programmable; the two latter are directly related to mathematics.

[56]Personal computers appear in 1977 and, against the predictions of the gurus, have taken up the scene, thanks no doubt to the impressive progress of hardware: a chip may contain at the end of the century up to 10^9 transistors

ability to actually calculate what before could only be imagined, and this works like an infection on the everyday practice of mathematics: mathematicians, scientists and engineers calculate orbits of satellites or trajectories of dynamical systems, numerical distributions or time series of real processes, weather maps or mathematical studies of singularities, temperature distributions in a furnace or statistical properties of the zeros of Riemann's Zeta function, ...

Among the most remarkable changes, Mathematics has had an important role in industrial and other applied processes in which laboratory experiments are combined with the new tools derived from Mathematics: there appears the combination of **mathematical modelization - mathematical and numerical analysis - simulation - visualization - control,** that forms a usual tool in the most diverse fields: communications, weather prediction, astrophysics, mining and the oil industry, ecology, industrial engineering, the car industry, economy and finance, and quite recently biology and medicine, as we will see with some detail in section 8. This interest gives rise to big institutes and computation centers all over the world. And new disciplines arise, like CFD, i.e., Computational Fluid Dynamics, or CB, Computational Biology.

The new concepts: numerical model, computer simulation, numerical experiment or exploration, dynamical visualization,... have become daily practice in scientific and industrial media. The development of methods of numerical formulation of the continuous models of physics, like differential and integral equations, is a fundamental branch of computational mathematics (viz, the methods of finite differences, finite elements[57], finite volumes,...). The study of the properties and convergence of these methods constitutes Numerical Analysis, that has a deep connection to Algebra. On the other hand, the computation capacity gives new life to the discrete mathematics, as graph theory, with its important applications (for example, to the telephone networks and in general to the world of communications).

In summary, a view has emerged where Computational Science is now the third leg of the scientific approach together with Theory and Experiment, and this view is nowadays strongly practiced in Physics, Chemistry and Engineering.

[57]Finite elements are a wonderful example of the development of a mathematical-numerical tool by the parallel but separate efforts of mathematicians and engineers, see an interesting historical account in [2]. The phenomenon is not isolated, cf. the recent history of wavelets. These examples should lead us to think a bit more about the benefits of communication.

8 Trends at the beginning of the XXIth century. Mathematics in Science, Industry and Business

We have seen the recent evolution of pure and applied mathematics towards theoretical consistency and universality of interests. In consonance with this, the panorama of current interests and future trends in the world of Mathematics offers an impressive variety. Using a somewhat rhetorical language, we may say that Mathematics is today *ubiquitous*, it is everywhere, and *relevant,* it matters. Mathematical modeling plays a bigger role than ever in science, engineering, business and the social sciences.

We will mention next some of the main applied topics as they appear in the literature, in conferences, in programs of major research institutes. We point out in italics related mathematical aspects for the reader's convenience.

• Celestial Mechanics. Problems of aeroespacial science. *Stability and chaos in dynamical systems. Strange attractors.* Mechanics of solids and fluids in zero gravity.

• Theory of fluids. Application to meteorology and climatology. Ocean engineering. Global warming and other geo-social issues. Complex Environmental Problems, global warming, environment engineering, *global circulation models, balance models.* Glaciology. Acoustics and application to the sound industry. Industrial fluids, lubrication. Turbulence. *Predictability and chaos. Stability, bifurcation. Free boundary problems. Hierarchies of approximate problems (as the geostrophic model).* New areas like fluid and structure interaction.

• Aeronautics. Hydrodynamical problems, supersonic and transonic flight. Airfoil design. Problems of combustion (flame propagation, detonation). *Shock waves and hyperbolic equations. Boundary Layers and asymptotic developments. Traveling Waves.*

• Modern Physics. The Mathematics of the atomic world and of elementary particles. The standard model, quantum electro-dynamics, quantum chromo-dynamics. *Group theory, renormalization and gauge theories, supersymmetry, Yang-Mills equations, instantons, dilatons, branes,... exotic geometries and topologies in higher dimensions.*

• Astrophysics. General relativity, stellar models. Mathematics of plasma physics, magnetohydrodynamics. *Kinetic equations (Boltzmann, Landau, Fokker-Planck, Vlasov, ...).*

• Geosciences. Problems of resources and mining. Complex Environmental Problems, global warming, environment engineering, *The equations of oil extraction, of groundwater filtration, of contaminant dispersal: nonlinear systems of PDEs and free boundary problems.* Mathematics of seismic phenomena, *wave propagation, inverse problems.*

• Material Science. Solid state physics. Nanotechnology. Coupling of quantum states, mesoscopic and continuous. *Semiclassical Boltzmann theory, Wigner equation.* Composites. *Linear and nonlinear Elasticity. Homogenization theory.* Fracture theory. Polymers. Superconductors.

• Strength of materials. Microstructures, composites, new materials. Theory of fracture. *Mathematical theories of calculus of variations and the homogenization. Singularity formation and evolution, blow-up problems.*

• Industrial Engineering. Steel industry, blast furnaces. Prototypes for the car industry (fluids, aerodynamics, materials and fracture theory).

• Electromagnetic field theory. Communications, antennae, radar. Industrial applications: microwave ovens couple Maxwell equations with Fourier heat theory.

• Discrete Mathematics. *Graph theory, combinatorics.*

• Computer Science. *Mathematical logic, algorithmia, computational complexity, parallelization.* Finite automata, formal languages, *algebra*. Machine learning, data mining, artificial intelligence, natural language processing.

The design of the quantum computer would open a new world to computation.

• Control. Optimal control, robust control, nonlinear control. Predictive control. Fuzzy control systems. Neural networks, Fault Detection and Diagnosis in Industrial Processes. Modeling and Control of Economic Systems. Constraint Based Scheduling. Communication and Control of Distributed Hybrid Systems.

• Automation and Robotics. *Algebraic geometry and computation.* Computer Vision and Virtual Reality. Biological and Computational Learning.

• Information theory. Coding of messages, error-correcting codes. Surprising applications of *number theory and algebra.* Image Processing and Compression. *Wavelets, fractals, nonlinear PDE theories.*

• Statistics in Science, Industry, Government and Business. Estimation and hypothesis testing, design of experiments. Reliabilitiy, survival analysis. Stochastic processes. Time series. Epidemiology. Quality control. Analysis of Variance. Multivariate analysis. Survey sampling, polls.

• Optimization Theory and Mathematical Programming. Integer Programming: Facets, Subadditivity, and Duality. Nonlinear programming, convex programming. Interative Methods. Industrial Design Optimization. *Numerical methods, partial differential equations, calculus of variations, combinatorics, linear algebra.*

• Problems of optimal transportation. Problems of traffic. Traffic in the *Web.*

• Economy. Financial calculus unites *stochastic differential equations, partial differential equations and free boundary problems.* Models for the global economy.

• Chemistry. Quantum Chemistry: *simulation of atomic and molecular structures* *through fundamental equations.* Reaction dynamics, combustion. *Mathematics of nucleation, growth of crystals and chemotaxis. Front propagation, traveling waves, chemical oscillators. Chaos.* Drug design.

Life Sciences and Medicine:

• Biology: Population models. Mathematics of Genetics, Computational phylogenetics. Nucleic Acid structure and function. Molecular evolution. Proteomics. Regulatory and developmental pathway inference ADN computation.

• Medicine: interaction fluid-structure as a model for the blood flow. Modelization and simulation of the function of other organs: brain, lungs and liver. *Self-organization and fractal geometries.* Computational neuroscience. The Mathematics of infectious diseases.

• Tomography. Computerized tomography, 3D image reconstruction. Tumor growth models. *Fourier and Radon Transforms.*

• Though Computational Mathematics (as different from Computer Science) permeates all fields of application, it derserves a mention in itself: numerical methods and codes; efficient algorithms; approximation, (a priori and a posteriori) error estimates, adaptive methods and adaptive models, ...

• On the other hand, Mathematical Modeling in its different variants (deterministic, continuous, discrete, ...) leads to the problems of Model Validation and the techniques of obtention and elaboration of data on which validation is based (see Statistics above), as well as the quite important (and debated) concept of hiereracy of models, a progressive way of approaching "reality" that is nowadays recognized and embedded into the toolkit of the applied scientist (the old idealist with their eternal truth will revolve in their graves; or will they not?).

We shall stop here and take a much needed break with some comments. The list is losely organized by affinity of topics; however, the close interconnection of the branches of applied mathematics forces us to indulge in repetitions, or otherwise to place a subject under one of various possible headings. On the other hand, we are leaving without proper comment a number of fields of application: the theory of complex systems, selfsimilarity in the natural world, pattern formation and recognition, the global positioning systems (GPS), mathematics of electoral systems, Architecture or the food industry. And there is the trend for Mathematics to play an important role in the Arts, as it already does in the Entertainment Industry combined with the formidable progress of computer technology. And how could I forget talking to you about Knot theory, the Simplex Method or the Kalman Filter? In conclusion, this long list is incomplete because of the limited knowledge of the author, but I hope that it will impress upon the reader the enormous variety of interests of today's applied mathematics.

I would like to add a final personal reflection on the trends I see underlining all the above diversity. The mathematics that are to come will be much more **stochastic** and **algorithmic** than they used to be in the XXth century, and **mathematical modeling** will come to be considered an essential part of the mathematical education and activity. But whatever happens, it looks to me that a clear and complete **proof**, and elegant if possible, will always be the heart of the matter, as it has been since good old Euclid, and future mathematicians will still get excitement from **problems and conjectures,** and as Galileo did, from **looking at the world** (or the stars). And they will build, perched on the shoulders of former giants, these delicate, intricate and elusive objects called **theories,** some of them destined to oblivion, some to eternity, or to the daily wear-and-tear. Who marvels anymore at the surprising existence of electromagnetic waves filling the air, now that they have even become a form of pollution? But so much for philosophy at this moment.

9 From Hilbert's 23 problems in 1900 to the Clay Millenium Problems in 2000

We have already pointed out the deep impact that the list of problems proposed by D. Hilbert in 1900 had upon his contemporaties and successors. One hundred years later different initiatives try to follow the example of the great man, see e.g. the books by Arnold-Atiyah-Lax-Mazur, and by Engquist-Schmid[58]. On wednesday May 24th, 2000 at the Collège of France in Paris the official announcement was made of the collection of seven mathematical problems that constitute the Millennium Prize Problems, sponsored by the Mathematics Clay Institute. Remembering Hilbert, it tries to reflect seven of the most important open problems of the mathematical science at the beginning of the new century[59]. These problems cover quite different areas of pure and applied mathematics. Here is the list

1. P versus NP (Computation theory)
2. The Hodge Conjecture (Algebraic geometry)
3. The Poincaré Conjecture (Geometry and topology)
4. The Riemann Hypothesis (Number theory)

[58] For more information see the article by A. Jackson cited in the final references. See also vol. 3, no. 1 (2000) of Gaceta de la Real Sociedad Matemática Española, article by J. L. Fernńdez and M. de León.

[59] the solution of each problem would mean for the author a prize of 1 millon of dollars. All the information about the prize and the problems can be obtained from the website `http: //www.claymath.org/prize_problems`.

5. Yang-Mills Existence and Mass Gap (Theoretical Physics)
6. Navier-Stokes Existence and Smoothness (Fluid Mechanics and PDEs)
7. The Birch and Swinnerton-Dyer Conjecture (Algebraic arithmetic geometry)

Let me add my personal mixed feelings about the list that seems destined to be famous and influential. Fortunately, it includes important open problems that cover varied topics of pure and applied mathematics. However, it does not do full justice to the vision of mathematics as the language and tool of science and engineering.

10 Examples of new courses

After two sections devoted to enumeration, it is time to take a closer look at some of the novelties of present-day mathematics. Among the many options the following three examples are taken from Finance, Communications and Fundamental Physics.

The Mathematics of financial uncertainty and risk

A remarkable example of the practical applications of Mathematics developed in the last decades is the so-called financial mathematics. The new financial instruments of *derivatives* are based on, and at the same time motivate this new branch of applied mathematics, which combines stochastic processes, partial differential equations and free boundary problems. The most famous result is the *Black-Scholes model*[60] for the option market, which reduces the pricing of a future to the solution of a heat equation. This reduction,I would like to record this reduction in the sequence of formulas

$$\frac{dS_t}{S_t} = \mu dt + \sigma\, dB_t \quad \Rightarrow \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + b\, S\, \frac{\partial P}{\partial S} - r\, P = 0,$$

which passes from a stochastic integral, representing random uncertainty, to a deterministic PDE, that allows for price valuation. This is a surprising example of *concept and technique transfer*, made possible by the common mathematical code (and by the fact that F. Black graduated in Quantum Physics). Since there is an inherent instability in those markets, and they have enormous consequences on the economy, both public and private, it is very important to try to apply mathematical methods to find the mathematical clue to the mechanisms that govern the evolution of such phenomena. This ia a real-world challenge for the new century.

From Fourier analysis to wavelets

[60] F. Black, M.Scholes, *The pricing of options and corporate liabilities*, 1973. Merton and Scholes received the Nobel Prize for Economy in 1997. A first version of the model had been proposed by Bachelier in 1900! it took seven decades for the application to occur.

We have discussed a while ago the problem faced by Fourier analysis when Du Bois Raymond proposed his example of nonconvergent Fourier series, and we want to recall here that the third option out of the problem consisted in changing the basis of functions used in the representation. This is what A. Haar did in 1909[61], thus solving the difficulty in principle, and we can say that this is the remote origin of wavelets, an idea that took a whole century to come of age. Prior to World War II investigation on the issue seems to have followed an exclusively mathematical interest with no application in mind whatsoever. But after the war engineers and applied scientists landed on the idea led by applications, notably in the information theory of Claude Shannon. Eventually the two strands merged and wavelet analysis has become an important intersection of the frontiers of mathematics, scientific computing and signal processing[62].

The mathematical models of Theoretical Physics

The two great scientific revolutions in XXth century Physics, i.e., Relativity and Quantum Mechanics, have impressed on the discipline a strong connection with pure mathematics and the enormous challenge of building a theory to unite both models into a consistent whole. Experimenters and theoreticians have taken up the quest for the "ultimate theory" which would explain all, from the constitution of the atom to the farthest recesses of the Universe. A final theory is still pending (and might be for a time) but great achievements have been obtained. Here are some milestones, all of them deep mathematics. Quantum Electrodymanics (QED) was developed to describe electromagnetic interaction in the framework of Quantum Mechanics, and deals with charges, photons and uses the beautiful Feynman diagrams. Next, Quantum Chromodynamics[63] does a similar job to describe the strong forces among *quarks*, the particles postulated by M. Gellmann and G. Zweig in 1964 as the building blocks of neutrons and protons. Out the four fundamental forces of Nature (gravitational, electromagnetic, weak and strong) the two intermediate have been given a unified theory in 1967 by S. Weinger, Sh. Glashow and Abdus Salam. *Symmetry*, *gauge* and *renormalization group* are keywords in this highly mathematical world. Maxwell's, Schrödinger's and Dirac's equations cede the place to Yang-Mills equations. The work crystallized in the early 70'sin the Standard Model of elementary particles, which explains atomic reality in terms of three generations of quarks and *leptons*. These particles interact through the $SU(2) \times U(1)$ theory for the electroweak force and the $SU(3)_{color}$ theory for the strong force. Mathematics is therefore at the core of the model, in the form of Lie groups, differential geometry (more specifically, connections on a fibre bundle) and partial differential equations.

[61] "Zur Theorie der orthogonalen Funktionensysteme", *Math. Annalen* **69** (1910), pp. 331-371

[62] Most of the data are taken from the book [15], cf. also [12]

[63] The name refers to the picturesque denomination for the conserved charge, called "color".

Grand Unified Gauge Theories attempt to combine both group theories into one. In String Theory the old basic idea of point particles is replaced by the idea of elementary vibrating strings. At the end of the century Superstring Theory proposes a mathematical model for the unification of all forces, hence of all physics. It lacks however sufficient experimental verification; without it a theory is just a theory. And the quest continues. These ideas have motivated quite important mathematical developments associated to names like Atiyah, Donaldson and Witten.

Physicists believe that the combination models-and-experiments will allow us to understand a strange world in which matter, space and time are not what we use to think, where empty space is full of activity and even there could exist many additional space dimensions curled up in ridiculously small distances (a typical distance would be 10^{-35}m, so that we do not see them, *voilà l'astuce*; but we see the mathematics, and in due time will see the consequences).

11 Facts and opinions

Not long ago there was a movement towards separation in Mathematics that seemed to move farther and farther away from each other the cultivators of both genres, pure and applied. And we should not forget the prejudice of many pure scientists against a type of applied mathematics more intent on profit than on scientific standards, and, on the other hand, the prejudice of many applied scientists towards the very artificial worlds of certain pure mathematics. Fortunately, we are witnessing a series of simultaneous events - namely, the explosion of the vitality of theoretical and computational mathematics, the successes of Mathematics in the formulation and solution of the key problems of contemporary Physics, Economy and Engineering, and the unsuspected variety of applications of all branches of Mathematics. These events are deeply modifying the vision of both fields, that tend to merge in one, in the best tradition of the past.

It is a great mystery for professionals that pure and applied mathematics work like the faces of the same coin. That they are not exactly the same is very well reflected in the words of Albert Einstein: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality"[64]. But the ideal and the practical meet with striking results. The amazement before the practical power of Mathematics is most vividly expressed by E. Wigner in a famous statement, where he wondered about "the unreasonable effectiveness of mathematics in the natural sciences"[65].

[64]From *Geometry and Science*, 1921. Included in *Sidelights of Relativity*, Dover, 1983.

[65]Conference in New York, 1959. Published in *Comm. Pure Applied Math.* **13** (1960), 1-14

A word has to be said about the changes in the way mathematics is being done, specially when it is applied. The emergence of the *computer era* has given mathematics new wings, *we can compute!* Efficient and fast computation has become available and cheap at the beginning of the XXIth century, and society needs more. Theorems will always be theorems and a logical derivation is the key to understanding, but the way to discovery will never be the same, and numerical performance is now central to most of mathematics (all of the applied mathematics). The effects on teaching will be no less drastic, but they are yet to be developed.

Another key feature of modern applied mathematics is *mathematical modeling*, the art of devising sensible *representations* of different phenomena of the real world in mathematical terms, based on rational *assumptions*. A model is only a model and reflects reality in the conflicting way that Einstein describes. But it is everything we have, unless we consider a better model (or even a hierarchy of them). This is the glory and the danger of modeling, a crucial aspect of today's applied mathematics. The current discussion about the predictions of mathematical climate models on global warming on the Earth show how important the issue is and how difficult is to manipulate partial evidence based on partial models and backed up by huge datasets of difficult interpretation. Which brings us back to the merit of the giant modelers like Newton, Maxwell, Einstein and the Quantum people.

As we have seen, a large part of the best Mathematics have originated to explain the working of the physical world. But it must be observed that very often the important consequences of Mathematics have not manifested immediately. The formulation of physical processes in mathematical key in the sense of Galileo requires a maturing process, and this process has its own rules and rhythm, that go from a few years to a few centuries. It would be a blessing if both the administration and education authorities were conscious of this fact in their decision-making.

On a more speculative level, the well-known mathematician and science writer Ian Stewart asserts that it is possible that Mathematics are efficient "because they represent the underlying language in the human brain". In that case we revert Galileo's bet in the sense that we maybe understand the world in mathematical terms because that is precisely the coding system in our mind. But this is a different debate.

Let me summarize some of the opinions I sustain for the sake of the everlasting debate:

- Only good Mathematics can be good Applied Mathematics. Applied Mathematics as an art which is different and separated from Mathematics as such, simply does not exist[66]. By putting Mathematics to use it changes them.

- Mathematics is really applied only if it answers an important problem of science,

[66] I take this forceful idea from A. Rényi, [26], who attributes it in the fiction to Archimedes.

technology, economy, or more generally, society;

- Though we can come to judge with a certain degree of reliability what is important today, the task of predicting what mathematics will be important in the long term (so-called strategic planning) exceeds the capacity of sensible persons, unless we simply answer in general terms like "good Mathematics will matter" or "the Mathematics of real-world problems will matter". Educated guesses and opinions on specific matters are human and may be useful as personal orientation, but when it comes to decisions and priorities prudence ought to be the rule.

So, are pure and applied mathematics the same after all? It is up to the reader to judge, but let me quote in a relaxed tone a saying of Yogi Berra[67]: "In theory, there is no difference between theory and practice; in practice, there is".[68]

12 Appendix on Mathematics in Spain

Spain played in a given moment of the late Middle Ages a significant role in the transmission of Arab culture to the West and there even existed a king in Seville who wrote poetry and promoted Mathematics. Al Andalus, the Arab Spain, had solid interests in the sciences, in particular medicine and astronomy, with fine scholars like Azarquiel of Toledo who composed astronomical tables. The Indian number system based on position was already in use in Al Andalus in the IX century. After its takeover by the Christians (1085 a.C.), Toledo, the city of three cultures -Christian, Arab and Jewish- was for centuries a main center of learning with its School of Translators, which brought into Latin the works of Greek and Arab authors. In another direction, the Majorcan Ramón Llull devised in his *Ars Magna* a whole art of algorithmic reasoning in which we can see the early precedents of the Boole algebra and the computer logic (Llull, who lived in the XIIIth century, is at the same time one of the oldest classics of the Catalan language). A century later nautical maps called *portulanos* from Majorca were the top of the art, and the names of Soler and Cresques are well-known. The latter, a Jew, participated in the organization of the Portuguese nautical school, which was at the origin of the discovery of the way to the Indies around Africa, and, indirectly, also of America.

But the medieval and early Reinassance hopes failed later in Spain, so that mathematics (and other sciences) have had a very humble history for centuries. While the

[67] famous American baseball player, well-known for his quips.

[68] Here is an amusing joke about the different views of mathematics: engineers think that the equations approximate reality; physicists think that reality approximates the equations; mathematicians are uncomfortable with the very idea of a connection between the equations and reality. Are we really?

Spanish literature and art stand at the peak of worldwide creation since the XVII th century and up to our time, it is apparent that no Spanish names appear in the famed textbooks of mathematical learning. There are in such texts numerous concepts and results named after authors belonging to the nations with a great scientific tradition: French, English, German, Italian, in more recent times Russian and American,.., as there are also frequent examples of other countries which due to their size and circumstances did not play such a prominent role in history but count in science. During these centuries of glorious development, let us say from Galileo to Einstein, Spanish names are not mentioned. Could history have been different? King Philip II realized the need for science and created a Mathematical Academy in Madrid (1582) under the direction of Juan de Herrera, the architect of El Escorial, but the institution did not take root and closed a few years later, while similar initiatives abroad gave birth to the Royal Society in England, the Académie de Sciences in France, and so on. There have been a number of brilliant isolated men, worthy of mention, like Pedro Ciruelo, Omerique, Jorge Juan and Echegaray, but a school never took root until very recently. For centuries Spanish students and professors were forbidden to travel and learn in foreign countries, a quite strong safety rule that prevented at the same time heterodoxy, science and progress.

This is not the place for a detailed study of History, for which we may refer to the specialists[69], so let me proceed by pointing out how we have recently come to a quite favorable present. Spain appeared to abandon its deep mathematical lethargy in the first half of the last century and the figure of Julio Rey Pastor can serve as a reference to a remarkable effort in making our country up-to-date, an effort based on a couple of main ideas: in the first place, by study abroad in the great foreign centers, and then by the import of the problems and topics that occupy the worldwide community. This method had a striking success in the development of North-American mathematics, and was having good results in our country in the first decades of the XXth century. However, our ill-fated history, and mainly in that respect the civil war, destroyed the effort, or in the best cases forced the scholars to exile, which then gave abundant fruit on Latin-American land, as is the case of well-known mathematicians like Luis Santaló and the bloomig of Argentinian mathematics. With a few very honourable exceptions, mathematical activity after the war and up to the 60s returned to the slumber of the past. Little by little began the awakening of Spain to normal mathematical life, specially in the 70s. After a decade of enormous effort of a generation that learnt from the original sources, taught from the most reliable textbooks in the classrooms, organized research seminars and traveled or sent their young students abroad, regular publication in recognized journals and participation in international events increased. In the 80's there came the decade of original creation, reflected in the number and

[69] like Juan Vernet, whose work [35] is used above

level of the publications in good journals. The signs of good times became many and unequivocal, and we may conclude that Spain is no longer different ("Spain is different" is a famous touristic motto dating from Franco's time, which had an obvious negative reading when applied to the troubles of Spain to become a modern country, and was therefore felt critically by the democratic opposition). The official indicators allow us to put figures to this evidence of change. From them we may deduce two facts which have initially surprised many:

(a) That Spanish mathematics have passed from a very modest place in 1980 (0.3% of the world production by the ISI Data Base) to a very honourable position at the moment, immediately after USA, Germany, England, France, Russia, Italy, Japan and Canada, with a production of relevant journals which has been multiplied by a factor of more than 10 and represents a worldwide more than 4.18% (ISI).

(b) That in the comparative outlook of Spanish science mathematics figures among the best placed specialties.

Another consequence of the creative state of Spanish mathematics is the presence of numerous and valuable textbooks, and research monographs in prestigious collections. Let it be said that Spain, which has reached a solid position in research, also counts on a tradition in mathematical education, with a very relevant role in ICMI.

Finally, the trend towards the computational and applied aspects of mathematics, with the emphasis on mathematics as the modeling tool per excellence, is now strongly felt in a community formerly very exclusively tied to pure mathematical thinking. Opening the windows to the wide world outside is an enormous challenge for the health of our mathematics and the wellfare of future generations, and all efforts are welcome. Let in the fresh air!

13 Conclusion

This is the end of our journey. Remembering Galileo, I would like to conclude as follows: the book of Nature is open before our eyes for us to admire in its infinite, changing and surprising beauty. Mathematics, as the language of Science, is here to help us understand it: besides, it may allow us to use it and exploit it, and this aspect is loaded with promises and dangers, as all human endeavours. I am confident that the mathematicians of today will make their contribution to understanding and improving the Information Society whose birth we have had the fortune to witness. In the era of computers and information *reality is in the number*, as Pythagoras would have liked. Or at least a big chunk of it.

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The Appendix reflects ideas of the author on the present of Spanish Mathematics taken with minor additions from reference [34], first section. More on the same subject in [33]. Interesting sources are Gaceta de la Real Soc. Matemática Española, vol. 3, 1 (2000) and Revista Española de Física, vol 14, no. 5, issues devoted to the state of Mathematics on the occasion of the World Mathematical Year celebration. See also [1, 5, 14, 20, 24, 31]. The list of references below reflects readings of the author while compiling this text and is not meant as a selection of the best reading on the subject.

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