Discontinuous Galerkin methods for Coupled Flow and Reactive Transport Problems[1]

Shuyu Sun and Mary F. Wheeler

The Center for Subsurface Modeling (CSM)
The Institute for Computational Engineering and Sciences (ICES)
The University of Texas, Austin, Texas 78712, USA

Abstract. Primal discontinuous Galerkin methods with interior penalty are proposed to solve the coupled system of flow and reactive transport in porous media, which arises from many applications including miscible displacement and acid stimulated flow. A cut-off operator is introduced in the discontinuous Galerkin schemes to treat the coupling of flow and transport and the coupling of transport and reaction. The uniform positive definitiveness and the uniform Lipschitz continuity are established for the commonly used dispersion/diffusion tensor. Interestingly, the polynomial degrees of approximation for the flow and the transport equations needs to be in the same order in order to maintain the convergence of DG applied to the coupled system. Optimal or nearly optimal convergences for both flow and transport are obtained when the same polynomial degrees of approximation are chosen for flow and transport. That is, error estimate in $L^2(H^1)$ for concentration is optimal in h and nearly optimal in p with a loss of 1/2; error estimates in semi-$L^\infty(H^1)$ for pressure and in $L^\infty(L^2)$ for velocity establish optimality in h and sub-optimality in p by 1/2; error estimates for concentration jump and pressure jump are optimal in both h and p.

1. Introduction

The discontinuous Galerkin (DG) methods [8, 22, 23, 40, 25, 26, 4, 5] have recently gained popularity for many attractive properties. First of all, the DG methods are locally mass conservative in the element level while most classical Galerkin finite element methods are not. In addition, they have less numerical diffusion than most conventional algorithms, thus are likely to offer more accurate solution, especially for advection-dominated transport problems. These methods are useful in treating rough coefficient problems and in capturing discontinuities in the solution due to the nature of employing discontinuous function spaces. The DG methods can naturally handle inhomogeneous boundary conditions and curved boundaries. The average of the trace of the fluxes from a DG solution along an element edge is continuous and can be extended so that a continuous flux is defined over the entire domain. As a consequence, DG can be easily coupled with conforming methods.

[1]**AMS subject classifications**: 65M15, 65M60, 65M50;

Key words and phrases: Discontinuous Galerkin methods, NIPG, IIPG, SIPG, *a priori* error estimates, coupled system, flow, transport, reaction;

Date: 11/20/2003.

Furthermore, for smooth flow and transport problems, DG with varying p can yield nearly exponential convergence rates. For time-dependent problems in particular, the mass matrices are block diagonal for DG, but not for conforming methods. This provides a computational advantage, especially if explicit time integrations are used.

The approximation spaces for DG are localized in each element, which provides a flexibility allowing for general non-conforming meshes with variable degree of approximation. This results in a substantially easier h-p adaptive implementation for DG than for conventional approaches. This flexibility also increases the efficiency in adaptivities because the unnecessary areas do not need to be refined in order to maintain conformity of the mesh. Moreover, DG has sharper error indicators available due to the localized behaviors of DG errors; in other words, there is less pollution of errors. This leads a more effective adaptivity for DG than for nonconforming methods. In addition, for time dependent transient problems, the nonconforming nature of DG allows for a easy and effective mesh modification dynamically with time. This dynamic adaptivity is crucial for massive transient problems involving a long simulation time, in particular, for problems where strong physics occurs in a small part of the domain with a moving location.

From a computer science point of view, the DG methods are easier to implement than most traditional finite element methods. The trial and test spaces are easier to construct than conforming methods because they are local. This results a simpler and more efficient implementation. For instance, DG methods are simpler to implement than two other locally conservative approaches, finite volume methods and mixed finite element methods. In particular, the implementation of finite volume methods for high order degree of approximation is substantially more difficult and less flexible than DG methods. The treatment of full tensor permeability or diffusivity usually demands an expanded form involving more computational efforts for mixed finite element methods, whereas DG methods naturally treat the full tensor due to its primal form. Unlike traditional finite element methods, the DG algorithms need only the mesh information about elements and interfaces, but without the mesh information about edges and vertices. Such a property of space dimension independence offers a great convenience to implement, test and debug the DG code. That is, we can rapidly debug and test the DG code in one space dimension, and then apply the same code to computational intensive three-dimensional problems. In addition, the simple communication pattern between elements makes DG potentially being well parallelizable, which is a necessity for many massive problems having excessive memory and CPU time requirements.

Flow and reactive transport are fundamental processes arising in many diversified fields such as petroleum engineering, groundwater hydrology, environmental engineering, soil mechanics, earth sciences, chemical engineering and biomedical engineering. Realistic simulations for simultaneous flow, transport and chemical reaction present significant computational challenges [2, 29, 45, 12, 17, 18, 21, 27, 28, 31, 37, 39, 19, 38, 46, 9, 32, 10, 20, 42], see also [11] and references therein. Traditional algorithms employ operator-splitting to treat flow, advection, diffusion-dispersion and chemical reaction sequentially and separately. Godunov [14] and characteristics [3, 15] are popular methods for the advection-diffusion subproblem. While the operator splitting approach allows one to employ different algorithms to each subproblem as well as to implement complicated kinetics in a modular fashion [41, 16, 29], it can result in slow convergence and a loss of accuracy [37, 41, 16]. DG has recently applied for flow and transport problems in porous media [43, 34, 24]. Four version of primal DG methods have been developed, namely, OBB-DG (Oden-Babuška-Baumann [22] DG scheme), NIPG (Non-symmetric Interior Penalty Galerkin) [24, 26], SIPG (Symmetric Interior Penalty Galerkin) [44, 33] and IIPG (Incomplete Interior Penalty Galerkin) [13, 33], for solution of flow and reactive transport problems. Explicit *a posteriori* error estimates of DG for reactive transport were studied in [36, 35]. DG for miscible displacement has been investigated by numerical experiments and was reported to exhibit good numerical performance [23]. However, to the best of our knowledge, the mathematical analysis on the convergence behavior of DG applied to coupled flow and transport problems has not been conducted. In this paper, we restrict our attention to the primal DG methods with interior penalty terms, i.e., NIPG, SIPG and IIPG. The estimates can be extended for the OBB-DG method.

The paper is organized as follows. In the following section, we describe the modeling equations. The DG schemes and some of their properties are introduced in section 3. Given the concentration error, the error estimates for the flow problem are derived in section 4. The error analyses for reactive transport problem are given in section 5 assuming the velocity error is known. Error estimates for the coupled system are obtained in section 6 based on the results in previous sections. The last section concludes with remarks.

2. Governing equations

In this paper, we consider coupled flow and reactive transport for a single flowing phase in porous media. Results for system of multiple species with kinetic reactions can be derived by similar arguments. For convenience, we will assume Ω is a polygonal and bounded domain in $\mathbb{R}^d (d = 1, 2 \text{ or } 3)$ with boundary $\partial\Omega = \overline{\Gamma}_{\text{in}} \cup \overline{\Gamma}_{\text{out}}$. Here we denote by Γ_{in} the inflow boundary and Γ_{out} the outflow/no-flow boundary, i.e.

$$\begin{aligned}\Gamma_{\text{in}} &= \{x \in \partial\Omega : \ \mathbf{u} \cdot \mathbf{n} < 0\}, \\ \Gamma_{\text{out}} &= \{x \in \partial\Omega : \ \mathbf{u} \cdot \mathbf{n} \geq 0\},\end{aligned}$$

where $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$. Let T be the final simulation time. The classical equations governing the flow and reactive transport in porous media are as follows.

- Flow equation

$$-\nabla \cdot (\mathbf{K}(c)\nabla p) \equiv \nabla \cdot \mathbf{u} = q, \ (x,t) \in \Omega \times (0,T], \tag{2.1}$$

- Reactive transport equation

$$\frac{\partial \phi c}{\partial t} + \nabla \cdot (\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c) = qc^* + r(c), \ (x,t) \in \Omega \times (0,T], \tag{2.2}$$

- Dispersion/diffusion tensor

$$\mathbf{D}(\mathbf{u}) = d_m \mathbf{I} + |\mathbf{u}| \left\{ \alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t \left(\mathbf{I} - \mathbf{E}(\mathbf{u}) \right) \right\}, \tag{2.3}$$

where the unknowns are p (the pressure in the fluid mixture), $\mathbf{u}$ (the Darcy velocity of the mixture, i.e. the volume of fluid flowing across a unit across-section per unit time) and c (the concentration of interested species, i.e., amount of species per unit volume of the fluid mixture). Here, we assume that the conductivity $\mathbf{K}$ is a globally Lipschitz continuous function of c, and is uniformly symmetric positive definite and bounded. The effective porosity ϕ is assumed to be time-independent, uniformly bounded above and below by positive numbers. The dispersion/diffusion tensor $\mathbf{D}(\mathbf{u})$ has contributions from molecular diffusion and mechanical dispersion, and can be calculated by equation (2.3), where $\mathbf{E}(\mathbf{u})$ is the tensor that projects onto the $\mathbf{u}$ direction, whose (i,j) component is $(\mathbf{E}(\mathbf{u}))_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}$; d_m is the molecular diffusivity and is assumed to be strictly positive; α_l and α_t are the longitudinal and transverse dispersivities, respectively, and are assumed to be nonnegative. The reaction term $r(c)$ is assumed to be a locally Lipschitz continuous function. The imposed external total flow rate q is a sum of sources (injection) and sinks (extraction) and is assumed to be bounded. Concentration c^* in the source term is the injected concentration c_w if $q \geq 0$ and is the resident concentration c if $q < 0$.

Flow and reactive transport are two-way coupled here. The velocity from the flow equation has a direct influence on the advection behavior of transport phenomena. On the other hand, the concentration from the transport equation affects conductivity, which has a significant influence on the flow pattern. The influence of conductivity by concentration can occur in many situations. For example, in miscible displacement, the viscosity is strongly affected by the concentration of species. The commonly used constitutive relation is the quarter-power mixing rule $\mu(c) = \left(c\mu_s^{-0.25} + (1-c)\mu_o^{-0.25}\right)^{-4}$. In acid stimulated flow, the permeability is dramatically affected by the reaction between chemical and rock.

We consider the following boundary conditions for this problem.

$$\mathbf{u}\cdot\mathbf{n} = u_B \qquad (x,t) \in \partial\Omega \times (0,T], \tag{2.4}$$

$$(\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c)\cdot\mathbf{n} = c_B \mathbf{u}\cdot\mathbf{n} \qquad (x,t) \in \Gamma_{\text{in}} \times (0,T], \tag{2.5}$$

$$(-\mathbf{D}(\mathbf{u})\nabla c)\cdot\mathbf{n} = 0 \qquad (x,t) \in \Gamma_{\text{out}} \times (0,T], \tag{2.6}$$

where c_B is the inflow concentration. The initial concentration is specified in the following way.

$$c(x,0) = c_0(x) \qquad x \in \Omega. \tag{2.7}$$

3. Discontinuous Galerkin scheme

3.1. **Notation.** Let $\mathcal{E}_h$ be a family of non-degenerate quasi-uniform and possibly non-conforming partitions of Ω composed of triangles or quadrilaterals if $d = 2$, or tetrahedra, prisms or hexahedra if $d = 3$. The non-degeneracy requirement (also called regularity) is that the element is convex, and that there exists $\rho > 0$ such that if h_j is the diameter of $E_j \in \mathcal{E}_h$, then each of the sub-triangles (for $d = 2$) or sub-tetrahedra (for $d = 3$) of element E_j contains a ball of radius ρh_j in its interior. The quasi-uniformity requirement is that there is $\tau > 0$ such that $\frac{h}{h_j} \leq \tau$ for all $E \in \mathcal{E}_h$, where h is the maximum diameter of all elements. We assume no element crosses the boundaries of Γ_{in} or Γ_{out}. The set of all interior edges (for 2 dimensional domain) or faces (for 3 dimensional domain) for $\mathcal{E}_h$ are denoted by Γ_h. On each edge or face $\gamma \in \Gamma_h$, a unit normal vector $\mathbf{n}_\gamma$ is chosen. The set of all edges or faces on Γ_{out} and on Γ_{in} for $\mathcal{E}_h$ are denoted by $\Gamma_{h,\text{out}}$ and $\Gamma_{h,\text{in}}$, respectively, for which the normal vector $\mathbf{n}_\gamma$ coincides with the outward unit normal vector.

For $s \geq 0$, we define,

$$H^s(\mathcal{E}_h) = \left\{ \phi \in L^2(\Omega) : \ \phi|_E \in H^s(E), \ E \in \mathcal{E}_h \right\}. \tag{3.1}$$

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h)$, $s > 1/2$. Let $E_i, E_j \in \mathcal{E}_h$ and $\gamma = \partial E_i \cap \partial E_j \in \Gamma_h$ with $\mathbf{n}_\gamma$ exterior to E_i. Denote

$$\{\phi\} = \frac{1}{2}\left(\left(\phi|_{E_i}\right)\Big|_\gamma + \left(\phi|_{E_j}\right)\Big|_\gamma \right), \tag{3.2}$$

$$[\phi] = \left(\phi|_{E_i}\right)\Big|_\gamma - \left(\phi|_{E_j}\right)\Big|_\gamma . \tag{3.3}$$

Denote the upwind value of concentration $c^*|_\gamma$ as follows:

$$c^*|_\gamma = \begin{cases} c|_{E_i} & \text{if } \mathbf{u}\cdot\mathbf{n}_\gamma \geq 0 \\ c|_{E_j} & \text{if } \mathbf{u}\cdot\mathbf{n}_\gamma < 0. \end{cases}$$

The usual Sobolev norm on Ω is denoted by $\|\cdot\|_{m,\Omega}$ [1]. The broken norms are defined, for positive integer m, as

$$|||\phi|||_m^2 = \sum_{E \in \mathcal{E}_h} \|\phi\|_{m,E}^2 . \tag{3.4}$$

The discontinuous finite element space is taken to be

$$\mathcal{D}_r\left(\mathcal{E}_h\right) \equiv \left\{ \phi \in L^2(\Omega) : \ \phi|_E \in \mathbb{P}_r(E), \ E \in \mathcal{E}_h \right\}, \tag{3.5}$$

where $\mathbb{P}_r(E)$ denotes the space of polynomials of (total) degree less than or equal to r on E. Note that we present error estimators in this paper for the local space $\mathbb{P}_r$, but the results also apply to the local space $\mathbb{Q}_r$ (the tensor product of the polynomial spaces of degree less than or equal to r in each space dimension) because $\mathbb{P}_r(E) \subset \mathbb{Q}_r(E)$.

The inner product in $\left(L^2(\Omega)\right)^d$ or $L^2(\Omega)$ is indicated by $(\cdot,\cdot)$ and the inner product in the boundary function space $L^2(\gamma)$ is indicated by $(\cdot,\cdot)_\gamma$. The norm $\left(L^p(\Omega)\right)^d$ for a vector-value function is defined as

$$\|\mathbf{u}\|_{(L^p(\Omega))^d} = \||\mathbf{u}|\|_{L^p(\Omega)} ,$$

where $|\cdot|$ is the standard vector norm defined by $|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{1/2}$. For simplicity, the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{(L^2(\Omega))^d}$ are also written as $\|\cdot\|_0$ for scalar-value and vector-value functions, respectively. The norm $\left(L^p(\Omega)\right)^{d\times d}$ for a matrix-value function is defined as

$$\|\mathbf{A}\|_{(L^p(\Omega))^{d\times d}} = \|\|\mathbf{A}\|_2\|_{L^p(\Omega)} ,$$

where $\|\cdot\|_2$ is the matrix 2-norm defined by $\|\mathbf{A}\|_2 = \sup_{|\mathbf{u}|=1} |\mathbf{A}\mathbf{u}|$. The cut-off operator $\mathcal{M}$ is defined as

$$\mathcal{M}(c)(x) = \min\left(c(x), M\right), \tag{3.6}$$

$$\mathcal{M}(\mathbf{u})(x) = \begin{cases} \mathbf{u}(x) & \text{if } |\mathbf{u}(x)| \leqslant M \\ M\mathbf{u}(x)/|\mathbf{u}(x)| & \text{if } |\mathbf{u}(x)| > M \end{cases}, \tag{3.7}$$

where M is a large positive constant. By a straightforward argument, we can show that the cut-off operator $\mathcal{M}$ is uniformly Lipschitz continuous in the following sense.

Lemma 3.1. (Property of operator $\mathcal{M}$) *The cut-off operator $\mathcal{M}$ defined as in equations (3.6) and (3.7) is uniformly Lipschitz continuous with a Lipschitz constant one, that is,*

$$\begin{aligned}\|\mathcal{M}(c)-\mathcal{M}(w)\|_{L^\infty(\Omega)} &\leq \|c-w\|_{L^\infty(\Omega)} \quad \forall c\in L^\infty(\Omega)\ \forall w\in L^\infty(\Omega),\\ \|\mathcal{M}(\mathbf{u})-\mathcal{M}(\mathbf{v})\|_{(L^\infty(\Omega))^d} &\leq \|\mathbf{u}-\mathbf{v}\|_{(L^\infty(\Omega))^d} \quad \forall \mathbf{u}\in (L^\infty(\Omega))^d\ \forall \mathbf{v}\in (L^\infty(\Omega))^d.\end{aligned}$$

We use the following hp approximation results, which can be proved using the techniques in [6, 7]. Let $E\in\mathcal{E}_h$, $\phi\in H^s(E)$ and h_E is the diameter of E. Then there exists a constant K independent of ϕ, r, and h_E and a sequence of $z_r^{h_E}\in P_r(E)$, $r=1,2,\cdots$, such that

$$\left\{\begin{array}{ll}\left\|\phi-z_r^{h_E}\right\|_{q,E}\leq K\frac{h_E^{\mu-q}}{r^{s-q}}\|\phi\|_{s,E} & 0\leq q<s,\\ \left\|\phi-z_r^{h_E}\right\|_{q,\partial E}\leq K\frac{h_E^{\mu-q-\frac{1}{2}}}{r^{s-q-\frac{1}{2}}}\|\phi\|_{s,E} & 0<q+\frac{1}{2}<s,\end{array}\right. \tag{3.8}$$

where $\mu=\min(r+1,s)$.

We shall also use the following inverse inequalities, which can be derived using the method in [30]. Let $E\in\mathcal{E}_h$, $v\in\mathbb{P}_r(E)$ and h_E is the diameter of E. Then there exists a constant K independent of v, r and h_E, such that

$$\left\{\begin{array}{ll}\|D^q v\|_{0,\partial E}\leq K\frac{r}{h_E^{1/2}}\|D^q v\|_E, & q\geq 0\\ \left\|D^{q+1}v\right\|_{0,E}\leq K\frac{r^2}{h_E}\|D^q v\|_{0,E} & q\geq 0.\end{array}\right. \tag{3.9}$$

3.2. **Continuous in time scheme.** We consider NIPG (the non-symmetric interior penalty Galerkin method), SIPG (the symmetric interior penalty Galerkin method) and IIPG (the incomplete interior penalty Galerkin method) for the flow and the transport equations. The three methods for flow and the three schemes for transport lead to nine different combinations for coupled flow and transport problems. However, we note that only IIPG for flow is compatible with primal DG methods for transport in the sense defined in [13].

For flow, we introduce the bilinear form $a(p,\psi)$ and the linear functional $l(\psi)$,

$$\begin{aligned}a(p,\psi;c) &= \sum_{E\in\mathcal{E}_h}\int_E \mathbf{K}(c)\nabla p\cdot\nabla\psi + J_{0,\text{flow}}(p,\psi)\\ &\quad -\sum_{\gamma\in\Gamma_h}\int_\gamma\{\mathbf{K}(c)\nabla p\cdot\mathbf{n}_\gamma\}[\psi] - s_{\text{flow}}\sum_{\gamma\in\Gamma_h}\int_\gamma\{\mathbf{K}(c)\nabla\psi\cdot\mathbf{n}_\gamma\}[p],\\ l(\psi) &= (q,\psi)-\sum_{\gamma\in\Gamma_{h,\text{out}}\cup\Gamma_{h,\text{out}}}\int_\gamma\psi u_B,\end{aligned}$$

where $s_{\text{flow}}=-1$ for NIPG, $s_{\text{flow}}=1$ for SIPG and $s_{\text{flow}}=0$ for IIPG. The interior penalty term $J_{0,\text{flow}}(p,\psi)$ for flow is defined as

$$J_{0,\text{flow}}(p,\psi)=\sum_{\gamma\in\Gamma_h}\frac{r_{\text{flow}}^2\sigma_{\gamma,\text{flow}}}{h_\gamma}\int_\gamma[p][\psi],$$

where the penalty parameter $\sigma_{\gamma,\text{flow}}$ is a constant on each edge or face γ. We assume $0<\sigma_{0,\text{flow}}\leq\sigma_{\gamma,\text{flow}}\leq\sigma_{m,\text{flow}}$.

For transport, we define the bilinear form $B(c, w; \mathbf{u})$ as

$$
\begin{aligned}
(3.10)\qquad & B(c,w;\mathbf{u}) \\
= & \sum_{E\in\mathcal{E}_h}\int_E (\mathbf{D}(\mathbf{u})\nabla c - c\mathbf{u})\cdot\nabla w - \int_\Omega c q^- w \\
& - \sum_{\gamma\in\Gamma_h}\int_\gamma \{\mathbf{D}(\mathbf{u})\nabla c\cdot\mathbf{n}_\gamma\}[w] - s_{\mathrm{transp}}\sum_{\gamma\in\Gamma_h}\int_\gamma \{\mathbf{D}(\mathbf{u})\nabla w\cdot\mathbf{n}_\gamma\}[c] \\
& + \sum_{\gamma\in\Gamma_h}\int_\gamma c^*\mathbf{u}\cdot\mathbf{n}_\gamma [w] + \sum_{\gamma\in\Gamma_{h,\mathrm{out}}}\int_\gamma c\mathbf{u}\cdot\mathbf{n}_\gamma w + J_{0,\mathrm{transp}}(c,w),
\end{aligned}
$$

where $s_{\mathrm{transp}} = -1$ for NIPG, $s_{\mathrm{transp}} = 1$ for SIPG and $s_{\mathrm{transp}} = 0$ for IIPG. The interior penalty term $J_{0,\mathrm{transp}}(p, \psi)$ for transport is defined as

$$
J_{0,\mathrm{transp}}(c,w) = \sum_{\gamma\in\Gamma_h}\frac{r_{\mathrm{transp}}^2\sigma_{\gamma,\mathrm{transp}}}{h_\gamma}\int_\gamma [c]\,[w],
$$

where the penalty parameter $\sigma_{\gamma,\mathrm{transp}}$ is a constant on each edge or face γ. We assume $0 < \sigma_{0,\mathrm{transp}} \le \sigma_{\gamma,\mathrm{transp}} \le \sigma_{m,\mathrm{transp}}$. Here q^+ is the injection source term and q^- is the extraction source term, i.e.,

$$
q^+ = \max(q,0); \quad q^- = \min(q,0).
$$

By definition, we have $q = q^+ + q^-$.

The linear functional $L(w; \mathbf{u}, c)$ is defined as

$$
(3.11)\qquad L(w;\mathbf{u},c) \;=\; \int_\Omega r(\mathcal{M}(c))\,w + \int_\Omega c_w q^+ w - \sum_{\gamma\in\Gamma_{h,\mathrm{in}}}\int_\gamma c_B\mathbf{u}\cdot\mathbf{n}_\gamma w.
$$

The continuous in time DG schemes for approximating (2.1)-(2.7) are as follows. We seek $P^{DG} \in W^{1,\infty}(0,T;\mathcal{D}_{r_{\mathrm{flow}}}(\mathcal{E}_h))$ and $C^{DG} \in W^{1,\infty}\left(0,T;\mathcal{D}_{r_{\mathrm{transp}}}(\mathcal{E}_h)\right)$ satisfying,

$$
(3.12)\quad a\left(P^{DG}, v; \mathcal{M}(C^{DG})\right) = l(v) \qquad \forall v\in\mathcal{D}_{r_{\mathrm{flow}}}(\mathcal{E}_h), \quad \forall t\in(0,T],
$$

$$
(3.13)\qquad \left(\frac{\partial\phi C^{DG}}{\partial t}, w\right) + B(C^{DG}, w; \mathbf{u}^M) = L(w;\mathbf{u}^M, C^{DG}),
$$

$$
\forall w\in\mathcal{D}_{r_{\mathrm{transp}}}(\mathcal{E}_h), \quad \forall t\in(0,T],
$$

$$
(3.14)\quad \left(\phi C^{DG}, w\right) = (\phi c_0, w), \qquad \forall w\in\mathcal{D}_{r_{\mathrm{transp}}}(\mathcal{E}_h), \quad t=0,
$$

where $\mathbf{u}^M \equiv \mathcal{M}(\mathbf{u}^{DG})$ with the DG velocity $\mathbf{u}^{DG}$ defined below.

$$
\begin{aligned}
(3.15)\qquad \mathbf{u}^{DG} &= -\mathbf{K}\left(\mathcal{M}(C^{DG})\right)\nabla P^{DG} \quad x\in E,\ E\in\mathcal{E}_h, \\
\mathbf{u}^{DG}\cdot\mathbf{n} &= -\left\{\mathbf{K}\left(\mathcal{M}(C^{DG})\right)\nabla P^{DG}\cdot\mathbf{n}\right\} \\
&\quad + \frac{r_{\mathrm{flow}}^2\sigma_{\gamma,\mathrm{flow}}}{h_\gamma}\int_\gamma\left(P^{DG}\big|_E - P^{DG}\big|_{\Omega\setminus\overline{E}}\right) \\
&\quad x\in\gamma = \partial E_i\cap\partial E_j,\ E_i, E_j\in\mathcal{E}_h \text{ and } \mathbf{n} \text{ exterior to } E_i, \\
\mathbf{u}^{DG}\cdot\mathbf{n} &= u_B \quad x\in\partial\Omega.
\end{aligned}
$$

Here, $\mathbf{u}^{DG}$ is defined at every interior point in each element, but only the normal velocity component $\mathbf{u}^{DG}\cdot\mathbf{n}$ is defined on element interfaces and on domain boundaries, because this is all the information needed in the DG schemes in the transport part.

3.3. **Some properties of DG.** The DG schemes are consistent. That is, the true solution, if existed and essentially bounded, satisfies the DG schemes. This is stated in the following lemma, noting that $\mathcal{D}_{r_{\text{flow}}}(\mathcal{E}_h) \subset H^{s_{\text{flow}}}(\mathcal{E}_h), s_{\text{flow}} > \frac{3}{2}$ and $\mathcal{D}_{r_{\text{transp}}}(\mathcal{E}_h) \subset H^{s_{\text{transp}}}(\mathcal{E}_h), s_{\text{transp}} > \frac{3}{2}$.

Lemma 3.2. (Consistency) *If p, c and $\mathbf{u}$ are the solution of (2.1)-(2.7) and are essentially bounded, then they satisfy*

$$a(p,v;c) = l(v) \qquad \forall v \in H^{s_{\text{flow}}}(\mathcal{E}_h), s_{\text{flow}} > \frac{3}{2}, \quad \forall t\in(0,T], \tag{3.16}$$

$$\left(\frac{\partial \phi c}{\partial t}, w\right) + B(c,w;\mathbf{u}) = L(w;\mathbf{u},c) \qquad \forall w \in H^{s_{\text{transp}}}(\mathcal{E}_h), s_{\text{transp}} > \frac{3}{2} \quad \forall t\in(0,T] \tag{3.17}$$

provided that the constant M for the cut-off operator is sufficiently large.

Proof. Let $w \in H^{s_{\text{transp}}}(\mathcal{E}_h)$, $s_{\text{transp}} > \frac{3}{2}$ and $E \in \mathcal{E}_h$. Multiplying equation (2.2) by $w|_E$, and integrating by parts, we have

$$\left(\frac{\partial \phi c}{\partial t}, w\right)_E - \int_E (\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c)\cdot\nabla w + \int_{\partial E}(\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c)\cdot \mathbf{n}_{\partial E} w = \int_E qc^* w + r(c)w \qquad (x,t)\in\Omega\times(0,T].$$

Summing over all elements in $\mathcal{E}_h$, noting that the trace of the concentration and its normal flux are continuous across edges/faces, and applying the boundary conditions, we obtain the result for transport. The result for flow follows by a similar argument. □

The element-wise mass conservative property of DG scheme is described as the following Lemma.

Lemma 3.3. (Local mass balance) *The approximation of the Darcy velocity satisfies on each element E the following local mass balance property for overall fluid.*

$$\begin{aligned} &\int_{\partial E\setminus\partial\Omega} \mathbf{u}^{DG}\cdot\mathbf{n} \\ \equiv\ & -\int_{\partial E\setminus\partial\Omega}\left\{\mathbf{K}\nabla P^{DG}\cdot\mathbf{n}\right\} \\ & + \sum_{\gamma\subset\partial E\setminus\partial\Omega}\frac{r_{\text{flow}}^2\sigma_{\gamma,\text{flow}}}{h_\gamma}\int_\gamma\left(P^{DG}\big|_E - P^{DG}\big|_{\Omega\setminus\overline{E}}\right) \\ =\ & \int_E q - \int_{\partial E\cap\partial\Omega} u_B \end{aligned} \tag{3.18}$$

The approximation of the concentration satisfies on each element E the following conservative property for the mass of the species.

$$\begin{aligned}
&\int_E \frac{\partial \phi C^{DG}}{\partial t} - \int_{\partial E \setminus \partial\Omega} \left\{ \mathbf{D}(\mathbf{u}^M) \nabla C^{DG} \cdot \mathbf{n} \right\} + \int_{\partial E} C^{DG*} \mathbf{u}^M \cdot \mathbf{n} \\
&\quad + \sum_{\gamma \subset \partial E \setminus \partial\Omega} \frac{r_{\text{transp}}^2 \sigma_{\gamma,\text{transp}}}{h_\gamma} \int_\gamma \left(C^{DG}\big|_E - C^{DG}\big|_{\Omega\setminus\overline{E}} \right) \\
&\qquad = \int_E C^{DG*} q + \int_E r\left(\mathcal{M}(C^{DG})\right).
\end{aligned} \tag{3.19}$$

Proof. The chemical mass balance relationship given in (3.19) follows from the DG scheme by fixing an element E and letting $w \in \mathcal{D}_{r_{\text{transp}}}(\mathcal{E}_h)$ with $w|_E = 1$, $w|_{\Omega\setminus E} = 0$. The mass balance relationship for overall fluid (3.18) follows from the DG scheme by fixing an element E and letting $v \in \mathcal{D}_{r_{\text{flow}}}(\mathcal{E}_h)$ with $v|_E = 1$, $v|_{\Omega\setminus E} = 0$. □

We remark that the ∂E terms in (3.18) and in (3.19) can be extended to a continuous flux defined over the entire domain Ω. We also note that the definition of DG velocity $\mathbf{u}^{DG}$ in (3.15) is compatible with the mass balance relationship (3.18).

4. Flow problem

Throughout the paper, we denote by K a generic positive constant that is independent of h and r, but might depend on the solution of PDE; we denote by ϵ an fixed positive constant that can be chosen arbitrarily small.

Theorem 4.1. (Error estimate for pressure) *Let $(\mathbf{u}, p, c)$ be the solution to (2.1)-(2.7), and assume $p \in L^2(0,T; H^{s_{\text{flow}}}(\mathcal{E}_h))$, $\mathbf{u} \in \left(L^2\left(0,T; H^{s_{\text{flow}}-1}(\mathcal{E}_h)\right)\right)^d$ and $c \in L^2(0,T; H^{s_{\text{transp}}}(\mathcal{E}_h))$. We further assume that p, ∇p, c and ∇c are essentially bounded. If the constant M for the cut-off operator and the penalty parameters are sufficiently large, then there exists a constant K independent of h and r such that*

$$\begin{aligned}
&\left\| \mathbf{K}^{1/2}(c) \nabla \left(P^{DG} - p\right) \right\|_0^2 (t) \\
&\leq K \left(1 + \frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right) \left\| c - C^{DG} \right\|_0^2 (t) + K \frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}} + K \frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}}
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&J_{0,\text{flow}}\left(P^{DG} - p, P^{DG} - p\right)(t) \\
&\leq K \left(1 + \frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right) \left\| c - C^{DG} \right\|_0^2 (t) + K \frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}} + K \frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-3}},
\end{aligned} \tag{4.2}$$

where $\mu_{\text{flow}} = \min(r_{\text{flow}} + 1, s_{\text{flow}})$, $\mu_{\text{transp}} = \min(r_{\text{transp}} + 1, s_{\text{transp}})$, $r_{\text{flow}} \geq 1$, $s_{\text{flow}} \geq 2$, $r_{\text{transp}} \geq 1$, $s_{\text{transp}} \geq 2$, and $\delta = 0$ in the cases of conforming meshes with triangles or tetrahedra. In general cases, $\delta = 1$.

Proof. Let $\widehat{p} \in \mathcal{D}_{r_{\text{flow}}}(\mathcal{E}_h)$ be an interpolant of concentration p such that the hp result (3.8) holds. Define

$$\begin{aligned} \xi &= P^{DG} - p, \\ \xi^I &= p - \widehat{p}, \\ \xi^A &= P^{DG} - \widehat{p} = \xi + \xi^I. \end{aligned} \tag{4.3}$$

Subtracting the DG scheme equation from the weak formulation, we have for any $w \in \mathcal{D}_r(\mathcal{E}_h)$,

$$a(\xi, v; \mathcal{M}(C^{DG})) + a(p, v; \mathcal{M}(C^{DG})) - a(p, v; c) = 0.$$

Splitting ξ according $\xi = \xi^A - \xi^I$, we have

$$\begin{aligned} & a(\xi^A, v; \mathcal{M}(C^{DG})) \\ = \ & a(p, v; c) - a(p, v; \mathcal{M}(C^{DG})) + a(\xi^I, v; \mathcal{M}(C^{DG})). \end{aligned} \tag{4.4}$$

Choosing $v = \xi^A$, we obtain

$$\begin{aligned} & a(\xi^A, \xi^A; \mathcal{M}(C^{DG})) \\ = \ & a(p, \xi^A; c) - a(p, \xi^A; \mathcal{M}(C^{DG})) + a(\xi^I, \xi^A; \mathcal{M}(C^{DG})). \end{aligned} \tag{4.5}$$

Let us first consider the left hand side of error equation (4.5).

$$\begin{aligned} & a(\xi^A, \xi^A; \mathcal{M}(C^{DG})) \\ = \ & \sum_{E \in \mathcal{E}_h} \int_E \mathbf{K}(\mathcal{M}(C^{DG})) \nabla \xi^A \cdot \nabla \xi^A + J_{0,\text{flow}}\left(\xi^A, \xi^A\right) \\ & -(1 + s_{\text{flow}}) \sum_{\gamma \in \Gamma_h} \int_\gamma \left\{\mathbf{K}(\mathcal{M}(C^{DG})) \nabla \xi^A \cdot \mathbf{n}_\gamma\right\} \left[\xi^A\right] \end{aligned}$$

The first two terms in above equation is nonnegative, and the third term can be bounded by

$$\begin{aligned} & \left| (1 + s_{\text{flow}}) \sum_{\gamma \in \Gamma_h} \int_\gamma \left\{\mathbf{K}(\mathcal{M}(C^{DG})) \nabla \xi^A \cdot \mathbf{n}_\gamma\right\} \left[\xi^A\right] \right| \\ \leq \ & \frac{h}{K r_{\text{flow}}^2} \sum_{E \in \mathcal{E}_h} \left\| \mathbf{K}^{\frac{1}{2}} \left(\mathcal{M}(C^{DG})\right) \nabla \xi^A \cdot \mathbf{n} \right\|_{0,\partial E}^2 + \frac{K r_{\text{flow}}^2}{h} \sum_{\gamma \in \Gamma_h} \left\| \left[\xi^A\right] \right\|_{0,\gamma}^2 \\ \leq \ & \frac{1}{2} \|\mathbf{K}^{\frac{1}{2}} \left(\mathcal{M}(C^{DG})\right) \nabla \xi^A \|_0^2 + \frac{1}{2} J_{0,\text{flow}}\left(\xi^A, \xi^A\right). \end{aligned}$$

where we have chosen penalty parameter to be sufficiently large such that $\sigma_{0,\text{flow}} \geq 2K$. We have,

$$\begin{aligned} & 2a(\xi^A, \xi^A; \mathcal{M}(C^{DG})) \\ \geq \ & \|\mathbf{K}^{\frac{1}{2}} \left(\mathcal{M}(C^{DG})\right) \nabla \xi^A \|_0^2 + J_{0,\text{flow}}\left(\xi^A, \xi^A\right). \end{aligned}$$

Let us bound the right hand side of error equation (4.5). The first two terms can be written as follows.

$$a(p, \xi^A; c) - a(p, \xi^A; \mathcal{M}(C^{DG}))$$

$$
\begin{aligned}
= & \sum_{E\in\mathcal{E}_h}\int_E \left(\mathbf{K}(\mathcal{M}(c))-\mathbf{K}(\mathcal{M}(C^{DG}))\right)\nabla p\cdot\nabla\xi^A \\
& -\sum_{\gamma\in\Gamma_h}\int_\gamma \left\{\left(\mathbf{K}(\mathcal{M}(c))-\mathbf{K}(\mathcal{M}(C^{DG}))\right)\nabla p\cdot\mathbf{n}_\gamma\right\}[\xi^A] \\
& -s_{\text{flow}}\sum_{\gamma\in\Gamma_h}\int_\gamma \left\{\left(\mathbf{K}(\mathcal{M}(c))-\mathbf{K}(\mathcal{M}(C^{DG}))\right)\nabla\xi^A\cdot\mathbf{n}_\gamma\right\}[p]
\end{aligned}
$$

Noting that $\left|\mathbf{K}(\mathcal{M}(c))-\mathbf{K}(\mathcal{M}(C^{DG}))\right|\leq K\left|c-C^{DG}\right|$ and $[p]=0$, we have

$$
\begin{aligned}
& \left|a(p,\xi^A;c)-a(p,\xi^A;\mathcal{M}(C^{DG}))\right| \\
\leq\ & K\left\|c-C^{DG}\right\|_0\left\|\nabla\xi^A\right\|_0 \\
& +K\sum_{E_i\cup E_j=\gamma\in\Gamma_h}\left(\int_\gamma\left|c-C^{DG}|_{E_i}\right|\left|[\xi^A]\right|+\int_\gamma\left|c-C^{DG}|_{E_j}\right|\left|[\xi^A]\right|\right) \\
\leq\ & K\left\|c-C^{DG}\right\|_0^2+\epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\
& +K\sum_{E_i\cup E_j=\gamma\in\Gamma_h}\frac{h_\gamma}{r_{\text{flow}}^2}\left(\int_\gamma\left|c-C^{DG}|_{E_i}\right|^2+\int_\gamma\left|c-C^{DG}|_{E_j}\right|^2\right) \\
\leq\ & K\left\|c-C^{DG}\right\|_0^2+\epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\
& +K\sum_{E_i\cup E_j=\gamma\in\Gamma_h}\frac{h_\gamma}{r_{\text{flow}}^2}\left(\int_\gamma\left|c-\widehat{c}|_{E_i}\right|^2+\int_\gamma\left|c-\widehat{c}|_{E_j}\right|^2\right) \\
& +K\sum_{E_i\cup E_j=\gamma\in\Gamma_h}\frac{h_\gamma}{r_{\text{flow}}^2}\left(\int_\gamma\left|\widehat{c}|_{E_i}-C^{DG}|_{E_i}\right|^2+\int_\gamma\left|\widehat{c}|_{E_j}-C^{DG}|_{E_j}\right|^2\right) \\
\leq\ & K\left\|c-C^{DG}\right\|_0^2+\epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\
& +K\frac{h}{r_{\text{flow}}^2}\frac{h^{2\mu_{\text{transp}}-1}}{r_{\text{transp}}^{2s_{\text{transp}}-1}}+K\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\sum_{E\in\mathcal{E}_h}\int_E\left|\widehat{c}-C^{DG}\right|^2 \\
\leq\ & K\left\|c-C^{DG}\right\|_0^2+\epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right)+K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-1}} \\
& +K\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\sum_{E\in\mathcal{E}_h}\int_E\left|\widehat{c}-c\right|^2+K\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\sum_{E\in\mathcal{E}_h}\int_E\left|c-C^{DG}\right|^2 \\
\leq\ & K\left(1+\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right)\left\|c-C^{DG}\right\|_0^2+\epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\
& +K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-1}}+K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}},
\end{aligned}
$$

where $\widehat{c}\in\mathcal{D}_{r_{\text{transp}}}(\mathcal{E}_h)$ is an interpolant of concentration c such that the hp result (3.8) holds and on the element interface $\gamma=E_i\cap E_j$, $\widehat{c}$ is defined as

$$
\widehat{c}|_\gamma=\{\widehat{c}\}=\frac{\widehat{c}|_{E_i}+\widehat{c}|_{E_j}}{2}.
$$

The third term in the right hand side of error equation (4.5) can be written as follows.

$$
\begin{aligned}
& a(\xi^I, \xi^A; \mathcal{M}(C^{DG})) \\
= & \sum_{E\in\mathcal{E}_h} \int_E \mathbf{K}(\mathcal{M}(C^{DG}))\nabla\xi^I \cdot \nabla\xi^A + J_{0,\text{flow}}\left(\xi^I, \xi^A\right) \\
& - \sum_{\gamma\in\Gamma_h} \int_\gamma \left\{\mathbf{K}(\mathcal{M}(C^{DG}))\nabla\xi^I \cdot \mathbf{n}_\gamma\right\}\left[\xi^A\right] \\
& - s_{\text{flow}} \sum_{\gamma\in\Gamma_h} \int_\gamma \left\{\mathbf{K}(\mathcal{M}(C^{DG}))\nabla\xi^A \cdot \mathbf{n}_\gamma\right\}\left[\xi^I\right] \\
:= & \sum_{i=1}^{4} T_i
\end{aligned}
$$

Term T_1 can be bounded using Cauchy-Schwartz inequality and approximation result,

$$
\begin{aligned}
|T_1| & \leq K \sum_{E\in\mathcal{E}_h} \int_E \left|\nabla\xi^I \cdot \nabla\xi^A\right| \\
& \leq \epsilon \left\|\nabla\xi^A\right\|_0^2 + K\left\|\nabla\xi^I\right\|_0^2 \\
& \leq \epsilon \left\|\nabla\xi^A\right\|_0^2 + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2}}.
\end{aligned}
$$

Term T_2 can be bounded using Cauchy-Schwartz inequality for penalty term,

$$
\begin{aligned}
|T_2| & \leq \epsilon J_{0,\text{flow}}\left(\xi^A, \xi^A\right) + K J_{0,\text{flow}}\left(\xi^I, \xi^I\right) \\
& \leq \epsilon J_{0,\text{flow}}\left(\xi^A, \xi^A\right) + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-3}}.
\end{aligned}
$$

Term T_3 can be bounded using approximation results on edge,

$$
\begin{aligned}
|T_3| & \leq \epsilon J_{0,\text{flow}}\left(\xi^A, \xi^A\right) + K\frac{h}{k^2}\sum_{\gamma\in\Gamma_h}\int_\gamma \left|\left\{\nabla\xi^I \cdot \mathbf{n}_\gamma\right\}\right|^2 \\
& \leq \epsilon J_{0,\text{flow}}\left(\xi^A, \xi^A\right) + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-1}}.
\end{aligned}
$$

Term T_4 can be bounded using the inverse inequality,

$$
\begin{aligned}
|T_4| & \leq K\sum_{\gamma\in\Gamma_h}\int_\gamma \left|\left\{\nabla\xi^A \cdot \mathbf{n}_\gamma\right\}\left[\xi^I\right]\right| \\
& \leq \epsilon\frac{h}{k^2}\sum_{\gamma\in\Gamma_h}\int_\gamma \left|\left\{\nabla\xi^A \cdot \mathbf{n}_\gamma\right\}\right|^2 + K\frac{k^2}{h}\sum_{\gamma\in\Gamma_h}\int_\gamma \left|\left[\xi^I\right]\right|^2 \\
& \leq \epsilon\left\|\nabla\xi^A\right\|_0^2 + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-3}}.
\end{aligned}
$$

When the mesh contains only triangles for two space dimensions or tetrahedra for three space dimensions and is conforming, we can choose a continuous interpolant, and then terms T_2 and T_4 vanish. Thus we have

$$\begin{aligned} \left|a(\xi^I,\xi^A;C^{DG})\right| &\le \sum_{i=1}^4 |T_i| \\ \le\ & \epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\ & +K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2}}+K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-1}}+\delta K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s-3}} \\ \le\ & \epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right)+K\frac{h^{2\mu_{\text{flow}}-2}}{r^{2s_{\text{flow}}-2-\delta}}, \end{aligned}$$

where $\delta=0$ in the cases of conforming meshes with triangles or tetrahedra. In general cases, $\delta=1$.

Substituting all these inequalities into equation (4.5), we have,

$$\begin{aligned} & \|\mathbf{K}^{\frac{1}{2}}\left(\mathcal{M}(C^{DG})\right)\nabla\xi^A\|_0^2+J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\ \le\ & K\left(1+\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right)\left\|c-C^{DG}\right\|_0^2+\epsilon\left\|\nabla\xi^A\right\|_0^2+\epsilon J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\ & +K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-1}}+K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}}+K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}}. \end{aligned}$$

Using the facts $\frac{1}{K}\mathbf{I}\le\mathbf{K}\left(\mathcal{M}(C^{DG})\right)\le K\mathbf{I}$ and $\frac{1}{K}\mathbf{I}\le\mathbf{K}\,(c)\le K\mathbf{I}$, we have

$$\begin{aligned} & \|\mathbf{K}^{\frac{1}{2}}\,(c)\,\nabla\xi^A\|_0^2+J_{0,\text{flow}}\left(\xi^A,\xi^A\right) \\ \le\ & K\left(1+\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right)\left\|c-C^{DG}\right\|_0^2 \\ & +K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}}+K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}}. \end{aligned}$$

The theorem follows from approximation results and the triangle inequality. □

Error estimates for velocity follow directly from the results on pressure.

Theorem 4.2. (Error estimate for velocity) *Let* $(\mathbf{u},p,c)$ *be the solution to (2.1)-(2.7), and assume* $p\in L^2\left(0,T;H^{s_{\text{flow}}}(\mathcal{E}_h)\right)$, $\mathbf{u}\in\left(L^2\left(0,T;H^{s_{\text{flow}}-1}(\mathcal{E}_h)\right)\right)^d$ *and* $c\in L^2\left(0,T;H^{s_{\text{transp}}}(\mathcal{E}_h)\right)$. *We further assume that* p, ∇p, c *and* ∇c *are essentially bounded. If the constant* M *for the cut-off operator and the penalty parameters are sufficiently large, then there exists a constant* K *independent of* h *and* r *such that*

$$\begin{aligned} \left\|\mathbf{u}^{DG}-\mathbf{u}\right\|_0^2(t) \ \le\ & K\left(1+\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right)\left\|C^{DG}-c\right\|_0^2(t) \\ & +K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}}+K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}}, \end{aligned} \tag{4.6}$$

and

$$\sum_{\gamma\in\Gamma_h} \left\|\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma - \mathbf{u}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t) \tag{4.7}$$
$$+ \sum_{E_i\cap E_j=\gamma\in\Gamma_h}\left(\left\|\mathbf{u}^{DG}\big|_{E_i} - \mathbf{u}\right\|^2_{0,\gamma}(t) + \left\|\mathbf{u}^{DG}\big|_{E_j} - \mathbf{u}\right\|^2_{0,\gamma}(t)\right)$$
$$\leq K\left(\frac{r^2_{\text{flow}}}{h} + \frac{r^2_{\text{transp}}}{h}\right)\left\|c - C^{DG}\right\|^2_0(t) + K\frac{h^{2\mu_{\text{transp}}-1}}{r^{2s_{\text{transp}}-2}_{\text{transp}}} + K\frac{h^{2\mu_{\text{flow}}-3}}{r^{2s_{\text{flow}}-4-\delta}_{\text{flow}}},$$

where $\mu_{\text{flow}} = \min(r_{\text{flow}}+1, s_{\text{flow}})$, $\mu_{\text{transp}} = \min(r_{\text{transp}}+1, s_{\text{transp}})$, $r_{\text{flow}} \geq 1$, $s_{\text{flow}} \geq 2$, $r_{\text{transp}} \geq 1$, $s_{\text{transp}} \geq 2$, *and* $\delta = 0$ *in the cases of conforming meshes with triangles or tetrahedra. In general cases,* $\delta = 1$.

Proof. The estimate relation (4.6) follows from Theorem 4.1 and the definition of DG velocity $\mathbf{u}^{DG} = -\mathbf{K}(\mathcal{M}(C^{DG}))\nabla P^{DG}$ at $x \in E$, $E \in \mathcal{E}_h$. To bound $\sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma - \mathbf{u}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t)$, we let $\widehat{P} \in \left(\mathcal{D}_{r_{\text{flow}}}(\mathcal{E}_h)\right)^d$ be an interpolant of p such that the hp result (3.8) holds. We define $\widehat{\mathbf{u}} = \mathbf{K}(\mathcal{M}(C^{DG}))\nabla\widehat{P}$ in each element. On the element interface $\gamma = E_i \cap E_j$, $\widehat{\mathbf{u}}$ is defined as

$$\widehat{\mathbf{u}}\big|_\gamma = \{\widehat{\mathbf{u}}\} = \frac{\widehat{\mathbf{u}}|_{E_i} + \widehat{\mathbf{u}}|_{E_j}}{2}.$$

Using the inverse inequality to relate values of $\mathbf{u}^{DG}$ on interfaces with those inside elements, we have

$$\sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma - \mathbf{u}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t)$$
$$\leq \sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma - \widehat{\mathbf{u}}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t) + \sum_{\gamma\in\Gamma_h}\left\|\widehat{\mathbf{u}}\cdot\mathbf{n}_\gamma - \mathbf{u}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t)$$
$$\leq \sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma - \widehat{\mathbf{u}}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t) + K\frac{h^{2\mu_{\text{flow}}-3}}{r^{2s_{\text{flow}}-3}_{\text{flow}}}$$
$$= \sum_{\gamma\in\Gamma_h}\left\|-\left\{\mathbf{K}(\mathcal{M}(C^{DG}))\nabla P^{DG}\cdot\mathbf{n}_\gamma\right\} + \frac{r^2_{\text{flow}}\sigma_{\gamma,\text{flow}}}{h_\gamma}\left[P^{DG}\right] - \widehat{\mathbf{u}}\cdot\mathbf{n}_\gamma\right\|^2_{0,\gamma}(t)$$
$$+K\frac{h^{2\mu_{\text{flow}}-3}}{r^{2s_{\text{flow}}-3}_{\text{flow}}}$$
$$\leq \frac{1}{2}\sum_{E_i\cup E_j=\gamma\in\Gamma_h}\left(\left\|\mathbf{u}^{DG}\big|_{E_i} - \widehat{\mathbf{u}}\big|_{E_i}\right\|^2_{0,\gamma}(t) + \left\|\mathbf{u}^{DG}\big|_{E_j} - \widehat{\mathbf{u}}\big|_{E_j}\right\|^2_{0,\gamma}(t)\right)$$
$$+\sum_{\gamma\in\Gamma_h}\left\|\frac{r^2_{\text{flow}}\sigma_{\gamma,\text{flow}}}{h_\gamma}\left[P^{DG}\right]\right\|^2_{0,\gamma}(t) + K\frac{h^{2\mu_{\text{flow}}-3}}{r^{2s_{\text{flow}}-3}_{\text{flow}}}$$
$$\leq K\frac{r^2_{\text{flow}}}{h}\sum_{E\in\mathcal{E}_h}\int_E\left|\mathbf{u}^{DG} - \widehat{\mathbf{u}}\right|^2 + J_{0,\text{flow}}\left(P^{DG}-p, P^{DG}-p\right)(t) + K\frac{h^{2\mu_{\text{flow}}-3}}{r^{2s_{\text{flow}}-3}_{\text{flow}}}$$
$$\leq K\frac{r^2_{\text{flow}}}{h}\sum_{E\in\mathcal{E}_h}\int_E\left|\mathbf{u}^{DG} - \mathbf{u}\right|^2 + K\frac{r^2_{\text{flow}}}{h}\sum_{E\in\mathcal{E}_h}\int_E\left|\mathbf{u} - \widehat{\mathbf{u}}\right|^2$$

$$
\begin{aligned}
&+J_{0,\text{flow}}\left(P^{DG}-p,P^{DG}-p\right)(t)+K\frac{h^{2\mu_{\text{flow}}-3}}{r_{\text{flow}}^{2s_{\text{flow}}-3}}\\
\leq\ & K\frac{r_{\text{flow}}^2}{h}\sum_{E\in\mathcal{E}_h}\int_E\left|\mathbf{u}^{DG}-\mathbf{u}\right|^2+K\frac{r_{\text{flow}}^2}{h}\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2}}\\
&+J_{0,\text{flow}}\left(P^{DG}-p,P^{DG}-p\right)(t)+K\frac{h^{2\mu_{\text{flow}}-3}}{r_{\text{flow}}^{2s_{\text{flow}}-3}}\\
\leq\ & K\frac{r_{\text{flow}}^2}{h}\sum_{E\in\mathcal{E}_h}\int_E\left|\mathbf{u}^{DG}-\mathbf{u}\right|^2+J_{0,\text{flow}}\left(P^{DG}-p,P^{DG}-p\right)(t)+K\frac{h^{2\mu_{\text{flow}}-3}}{r_{\text{flow}}^{2s_{\text{flow}}-4}}.
\end{aligned}
$$

Substituting (4.6) and (4.2) into above inequality, we bound the first term in (4.7). The second term in (4.7) follows by a similar argument. □

5. Reactive transport problem

We now state and prove two lemmas for the properties of dispersion/diffusion tensor, which will be used to prove the error estimates for the transport problem.

Lemma 5.1. (Uniform positive definiteness of $\mathbf{D}(\mathbf{u})$) *Let $\mathbf{D}(\mathbf{u})$ defined as in equation (2.3), where $d_m(x)\geq 0$, $\alpha_l(x)\geq 0$ and $\alpha_t(x)\geq 0$ are nonnegative functions of $x\in\Omega$. Then*

$$\mathbf{D}(\mathbf{u})\nabla c\cdot\nabla c\geq\left(d_m+\min\left(\alpha_l,\alpha_t\right)|\mathbf{u}|\right)|\nabla c|^2. \tag{5.1}$$

In particular, if $d_m(x)\geq d_{m,*}>0$ uniformly in the domain Ω, then $\mathbf{D}(\mathbf{u})$ is uniformly positive definite and for all $x\in\Omega$, we have,

$$\mathbf{D}(\mathbf{u})\nabla c\cdot\nabla c\geq d_{m,*}|\nabla c|^2. \tag{5.2}$$

Proof. Notice that

$$
\begin{aligned}
\mathbf{D}(\mathbf{u})\nabla c\cdot\nabla c &= d_m\nabla c\cdot\nabla c+|\mathbf{u}|\left\{\alpha_l\mathbf{E}(\mathbf{u})+\alpha_t\left(\mathbf{I}-\mathbf{E}(\mathbf{u})\right)\right\}\nabla c\cdot\nabla c\\
&= d_m|\nabla c|^2+|\mathbf{u}||\nabla c|^2\alpha_l\cos^2(\theta)+|\mathbf{u}||\nabla c|^2\alpha_t\left(1-\cos^2(\theta)\right)\\
&\geq \left(d_m+\min\left(\alpha_l,\alpha_t\right)|\mathbf{u}|\right)|\nabla c|^2,
\end{aligned}
$$

where θ is the angle between $\mathbf{u}$ and ∇c, i.e.

$$\cos(\theta)=\frac{\mathbf{u}\cdot\nabla c}{|\mathbf{u}||\nabla c|}.$$

□

Lemma 5.2. (Uniform Lipschitz continuousness of $\mathbf{D}(\mathbf{u})$) *Let $\mathbf{D}(\mathbf{u})$ defined as in equation (2.3), where, $d_m(x)\geq 0$, $\alpha_l(x)\geq 0$ and $\alpha_t(x)\geq 0$ are nonnegative of domain $x\in\Omega$, and the dispersivities α_l and α_t is uniformly bounded, i.e. $\alpha_l(x)\leq\alpha_l^*$ and $\alpha_t(x)\leq\alpha_t^*$. Then*

$$\left\|\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{v})\right\|_{(L^2(\Omega))^{d\times d}}\leq k_D\left\|\mathbf{u}-\mathbf{v}\right\|_{(L^2(\Omega))^d}, \tag{5.3}$$

where $k_D = (4\alpha_t^* + 3\alpha_l^*)\, d^{3/2}$ is a fixed number ($d = 2$ or 3 is the dimension of domain Ω).

Proof. Notice that

$$\begin{aligned}
& |\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_1 \\
= & \sum_{i=1}^{d} \max_{j=1,\cdots,d} \left|(\mathbf{D}(\mathbf{u}))_{i,j} - (\mathbf{D}(\mathbf{u}))_{i,j}\right| \\
= & \sum_{i=1}^{d} \max_{j=1,\cdots,d} \left|\alpha_t \delta_{ij} \left(|\mathbf{u}|_2 - |\mathbf{v}|_2\right) + (\alpha_l - \alpha_t)\left(\frac{u_i u_j}{|\mathbf{u}|_2} - \frac{v_i v_j}{|\mathbf{v}|_2}\right)\right| \\
\leq & \sum_{i=1}^{d} \max_{j=1,\cdots,d} \left|\alpha_t \delta_{ij} \left(|\mathbf{u}|_2 - |\mathbf{v}|_2\right)\right| + \sum_{i=1}^{d} \max_{j=1,\cdots,d} \left|(\alpha_l - \alpha_t)\left(\frac{u_i u_j}{|\mathbf{u}|_2} - \frac{u_i v_j}{|\mathbf{u}|_2}\right)\right| \\
& + \sum_{i=1}^{d} \max_{j=1,\cdots,d} \left|(\alpha_l - \alpha_t)\left(\frac{u_i v_j}{|\mathbf{u}|_2} - \frac{v_i v_j}{|\mathbf{u}|_2}\right)\right| \\
& + \sum_{i=1}^{d} \max_{j=1,\cdots,d} \left|(\alpha_l - \alpha_t)\left(\frac{v_i v_j}{|\mathbf{u}|_2} - \frac{v_i v_j}{|\mathbf{v}|_2}\right)\right| \\
\leq & \; d\alpha_t \left||\mathbf{u}|_2 - |\mathbf{v}|_2\right| + 3d \left|\alpha_l - \alpha_t\right| |\mathbf{u} - \mathbf{v}|_2 \\
\leq & \; \left(\alpha_t + 3\left|\alpha_l - \alpha_t\right|\right) d \,|\mathbf{u} - \mathbf{v}|_2 \,.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_2 & \leq \sqrt{d}\, |\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_1 \\
& \leq \left(\alpha_t + 3\left|\alpha_l - \alpha_t\right|\right) d^{3/2} \,|\mathbf{u} - \mathbf{v}|_2 \,,
\end{aligned}$$

where we have used the property of matrix norm: for any matrix $A \in \mathbb{R}^{m\times n}$,

$$\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\, \|A\|_1 \,.$$

The result follows by integration. □

We now present the error estimate for transport problem.

Theorem 5.3. (Error estimate for transport) *Let $(\mathbf{u}, p, c)$ be the solution to (2.1)-(2.7), and assume $p \in L^2(0,T; H^{s_{\text{flow}}}(\mathcal{E}_h))$, $\mathbf{u} \in \left(L^2\left(0,T; H^{s_{\text{flow}}-1}(\mathcal{E}_h)\right)\right)^d$, $c \in L^2(0,T; H^{s_{\text{transp}}}(\mathcal{E}_h))$, $\partial c/\partial t \in L^2\left(0,T; H^{s_{\text{transp}}-1}(\mathcal{E}_h)\right)$ and $c_0 \in H^{s_{\text{transp}}-1}(\mathcal{E}_h)$. We further assume that p, ∇p, c and ∇c are essentially bounded. If the constant M for the cut-off operator and the penalty parameters are sufficiently large, then there exists a constant K independent of h and r such that*

$$\begin{aligned}
(5.4) \quad & \left\|\sqrt{\phi}\left(C^{DG} - c\right)\right\|_0^2 (t) + \int_0^t \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u}) \nabla \left(C^{DG} - c\right)\|_0^2(\tau) d\tau \\
& + \int_0^t J_0^{\sigma}\left(C^{DG} - c, C^{DG} - c\right)(\tau) d\tau \\
\leq \; & K \int_0^t \left\|\sqrt{\phi}\left(C^{DG} - c\right)\right\|_0^2 (\tau) d\tau + K \int_0^t \left\|\mathbf{u} - \mathbf{u}^{DG}\right\|_0^2 (\tau) d\tau
\end{aligned}$$

$$+K\frac{h}{r_{\text{transp}}^2}\int_0^t \sum_{E_i\cap E_j=\gamma\in\Gamma_h}\left(\left\|\mathbf{u}^{DG}\big|_{E_i}-\mathbf{u}\right\|_{0,\gamma}^2+\left\|\mathbf{u}^{DG}\big|_{E_j}-\mathbf{u}\right\|_{0,\gamma}^2\right)(\tau)d\tau$$
$$+K\frac{h}{r_{\text{transp}}^2}\int_0^t \sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}\cdot\mathbf{n}_\gamma-\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma\right\|_{0,\gamma}^2(\tau)d\tau+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}}$$

where $\mu_{\text{flow}}=\min\left(r_{\text{flow}}+1,s_{\text{flow}}\right)$, $\mu_{\text{transp}}=\min\left(r_{\text{transp}}+1,s_{\text{transp}}\right)$, $r_{\text{flow}}\geq 1$, $s_{\text{flow}}\geq 2$, $r_{\text{transp}}\geq 1$, $s_{\text{transp}}\geq 2$, *and* $\delta=0$ *in the cases of conforming meshes with triangles or tetrahedra. In general cases,* $\delta=1$.

Proof. Let $\widehat{c}\in\mathcal{D}_{r_{\text{transp}}}(\mathcal{E}_h)$ be an interpolant of concentration c such that the hp result (3.8) holds. Define

$$\begin{aligned}\xi &= C^{DG}-c,\\ \xi^I &= c-\widehat{c},\\ \xi^A &= C^{DG}-\widehat{c}=\xi+\xi^I.\end{aligned}\tag{5.5}$$

Subtracting the DG scheme equation from the weak formulation, we have for any $w\in\mathcal{D}_{r_{\text{transp}}}(\mathcal{E}_h)$,

$$\begin{aligned}&\left(\frac{\partial\phi\xi}{\partial t},w\right)+B\left(\xi,w;\mathbf{u}^M\right)\\ &=L\left(w;\mathbf{u}^M,C^{DG}\right)-L\left(w;\mathbf{u},c\right)+B\left(c,w;\mathbf{u}\right)-B\left(c,w;\mathbf{u}^M\right).\end{aligned}$$

Splitting ξ according $\xi=\xi^A-\xi^I$, we have

$$\begin{aligned}&\left(\frac{\partial\phi\xi^A}{\partial t},w\right)+B(\xi^A,w;\mathbf{u}^M)\\ =\;& L\left(w;\mathbf{u}^M,C^{DG}\right)-L\left(w;\mathbf{u},c\right)+\left(\frac{\partial\phi\xi^I}{\partial t},w\right)+B(\xi^I,w;\mathbf{u}^M)\\ &+B\left(c,w;\mathbf{u}\right)-B\left(c,w;\mathbf{u}^M\right).\end{aligned}\tag{5.6}$$

Choosing $w=\xi^A$, we obtain

$$\begin{aligned}&\left(\frac{\partial\phi\xi^A}{\partial t},\xi^A\right)+B(\xi^A,\xi^A;\mathbf{u}^M)\\ =\;& L\left(\xi^A;\mathbf{u}^M,C^{DG}\right)-L\left(\xi^A;\mathbf{u},c\right)+\left(\frac{\partial\phi\xi^I}{\partial t},\xi^A\right)+B(\xi^I,\xi^A;\mathbf{u}^M)\\ &+B\left(c,\xi^A;\mathbf{u}\right)-B\left(c,\xi^A;\mathbf{u}^M\right).\end{aligned}\tag{5.7}$$

Let us first consider the left hand side of error equation (5.7).
The first term can be written as,

$$\left(\frac{\partial\phi\xi^A}{\partial t},\xi^A\right)=\frac{1}{2}\frac{d}{dt}\left\|\sqrt{\phi}\xi^A\right\|_{0,\Omega}^2.$$

The second term of equation (5.7) is,

$$\begin{aligned}&B(\xi^A,\xi^A;\mathbf{u}^M)\\ =\;&\sum_{E\in\mathcal{E}_h}\int_E\mathbf{D}(\mathbf{u}^M)\nabla\xi^A\cdot\nabla\xi^A-\sum_{E\in\mathcal{E}_h}\int_E\xi^A\mathbf{u}^M\cdot\nabla\xi^A-\int_\Omega q^-\left(\xi^A\right)^2\end{aligned}$$

$$
\begin{aligned}
& -(1+s_{\text{form}})\sum_{\gamma\in\Gamma_h}\int_\gamma \left\{\mathbf{D}(\mathbf{u}^M)\nabla\xi^A\cdot\mathbf{n}_\gamma\right\}\left[\xi^A\right] \\
& +\sum_{\gamma\in\Gamma_h}\int_\gamma \xi^{A*}\mathbf{u}^M\cdot\mathbf{n}_\gamma\left[\xi^A\right]+\sum_{\gamma\in\Gamma_{h,\text{out}}}\int_\gamma \mathbf{u}^M\cdot\mathbf{n}_\gamma\left(\xi^A\right)^2+J_{0,\text{transp}}\left(\xi^A,\xi^A\right) \\
=: \; & \sum_{i=1}^{7} R_i.
\end{aligned}
$$

We usually integrate by parts the advection term at this step if transport is not coupled with flow [33], which transfers a few terms into nonnegative terms. However, for transport coupled with flow, we cannot do so because the velocity here is approximated solution $\mathbf{u}^M$ rather than true velocity $\mathbf{u}$. It is easy to see that terms R_1, R_3, R_6 and R_7 are nonnegative. We need to bound terms R_2, R_4 and R_5 by $R_1+R_3+R_6+R_7$. Term R_2 can be bounded using the boundedness of $\mathbf{u}^M$:

$$
\begin{aligned}
|R_2| &\le \sum_{E\in\mathcal{E}_h}\int_E \left|\xi^A\right|\left|\nabla\xi^A\right| \\
&\le \epsilon\|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2+K\left\|\sqrt{\phi}\xi^A\right\|_0^2.
\end{aligned}
$$

Term R_4 is the term that leads the requirement of sufficient large penalties for SIPG and IIPG. It can be bounded using the inverse inequality and the fact $\frac{1}{K}\mathbf{D}(\mathbf{u})\le\mathbf{D}\left(\mathbf{u}^M\right)\le K\mathbf{D}(\mathbf{u})$.

$$
\begin{aligned}
|R_4| &\le \frac{h}{Kr_{\text{transp}}^2}\sum_{E\in\mathcal{E}_h}\left\|\mathbf{D}^{\frac{1}{2}}\left(\mathbf{u}^M\right)\nabla\xi^A\cdot\mathbf{n}\right\|_{0,\partial E}^2 \\
&\quad +\frac{Kr_{\text{transp}}^2}{h}\sum_{\gamma\in\Gamma_h}\left\|\left[\xi^A\right]\right\|_{0,\gamma}^2 \\
&\le \frac{1}{2}\|\mathbf{D}^{\frac{1}{2}}\left(\mathbf{u}^M\right)\nabla\xi^A\|_0^2+\frac{1}{2}J_{0,\text{transp}}\left(\xi^A,\xi^A\right) \\
&\le \frac{1}{2}\|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2+\frac{1}{2}J_{0,\text{transp}}\left(\xi^A,\xi^A\right).
\end{aligned} \tag{5.8}
$$

where we have chosen penalty parameter $\sigma_{0,\text{transp}}$ to be sufficiently large such that $\sigma_{0,\text{transp}}\ge 2K$.

Term R_5 is bounded by

$$
\begin{aligned}
|R_5| &\le K\frac{h}{r_{\text{transp}}^2}\sum_{E\in\mathcal{E}_h}\left\|\xi^{A*}\right\|_{0,\partial E}^2+\epsilon\frac{r_{\text{transp}}^2}{Kh}\sum_{\gamma\in\Gamma_h}\left\|\left[\xi^A\right]\right\|_{0,\gamma}^2 \\
&\le K\left\|\sqrt{\phi}\xi^A\right\|_0^2+\epsilon J_{0,\text{transp}}\left(\xi^A,\xi^A\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& B(\xi^A,\xi^A;\mathbf{u}^M) \\
\ge\; & \frac{1}{3}\|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2+\frac{1}{3}J_{0,\text{transp}}\left(\xi^A,\xi^A\right)-K\left\|\sqrt{\phi}\xi^A\right\|_0^2.
\end{aligned}
$$

Let us bound the right hand side of error equation (5.7). Noting that $\mathbf{u}^M\cdot\mathbf{n}=\mathbf{u}^{DG}\cdot\mathbf{n}=\mathbf{u}\cdot\mathbf{n}$ on the domain boundary if the constant M for the cut-off operator is sufficiently large, we can write the first two terms as,

$$L\left(\xi^A;\mathbf{u}^M,C^{DG}\right)-L\left(\xi^A;\mathbf{u},c\right)=\int_\Omega\left(r\left(\mathcal{M}(C^{DG})\right)-r\left(\mathcal{M}(c)\right)\right)\xi^A$$

Using Lemma 3.1, we have,

$$\begin{aligned}\left|L\left(\xi^A;\mathbf{u}^M,C^{DG}\right)-L\left(\xi^A;\mathbf{u},c\right)\right| &\le K\left\|\sqrt{\phi}\xi^A\right\|_0^2+K\left\|\xi^I\right\|_0^2\\ &\le K\left\|\sqrt{\phi}\xi^A\right\|_0^2+K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{transp}}^{2s_{\text{transp}}}}\|c\|_s^2.\end{aligned}$$

The third term in the right hand side of of error equation (5.7) can be bounded as

$$\begin{aligned} &\left|\left(\frac{\partial\phi\xi^I}{\partial t},\xi^A\right)\right|\le K\left\|\frac{\partial\xi^I}{\partial t}\right\|_0\left\|\sqrt{\phi}\xi^A\right\|_0\\ \le\ &K\left\|\sqrt{\phi}\xi^A\right\|_0^2+K\left\|\frac{\partial\xi^I}{\partial t}\right\|_0^2\le K\left\|\sqrt{\phi}\xi^A\right\|_0^2+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-2}}\|c_t\|_{s-1}^2.\end{aligned}$$

The fourth term in the right hand side of of error equation (5.7) is

$$\begin{aligned} &B(\xi^I,\xi^A;\mathbf{u}^M)\\ =\ &\sum_{E\in\mathcal{E}_h}\int_E\mathbf{D}(\mathbf{u}^M)\nabla\xi^I\cdot\nabla\xi^A-\sum_{E\in\mathcal{E}_h}\int_E\xi^I\mathbf{u}^M\cdot\nabla\xi^A-\int_\Omega q^-\xi^I\xi^A\\ &-\sum_{\gamma\in\Gamma_h}\int_\gamma\left\{\mathbf{D}(\mathbf{u}^M)\nabla\xi^I\cdot\mathbf{n}_\gamma\right\}\left[\xi^A\right]-s_{\text{form}}\sum_{\gamma\in\Gamma_h}\int_\gamma\left\{\mathbf{D}(\mathbf{u}^M)\nabla\xi^A\cdot\mathbf{n}_\gamma\right\}\left[\xi^I\right]\\ &+\sum_{\gamma\in\Gamma_h}\int_\gamma\xi^{I*}\mathbf{u}^M\cdot\mathbf{n}_\gamma\left[\xi^A\right]+\sum_{\gamma\in\Gamma_{h,\text{out}}}\int_\gamma\mathbf{u}^M\cdot\mathbf{n}_\gamma\xi^I\xi^A+J_0^\sigma\left(\xi^I,\xi^A\right)\\ =:\ &\sum_{i=1}^8 T_i.\end{aligned}$$

Terms T_1 through T_3 can be bounded by using Cauchy-Schwartz inequality and approximation results,

$$\begin{aligned}|T_1| &\le \epsilon\|\mathbf{D}^{\frac12}\left(\mathbf{u}\right)\nabla\xi^A\|_0^2+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-2}}\|c\|_s^2,\\ |T_2| &\le \epsilon\|\mathbf{D}^{\frac12}\left(\mathbf{u}\right)\nabla\xi^A\|_0^2+K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{transp}}^{2s_{\text{transp}}}}\|c\|_s^2,\\ |T_3| &\le K\left\|\sqrt{\phi}\xi^A\right\|_0^2+K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{transp}}^{2s_{\text{transp}}}}\|c\|_s^2.\end{aligned}$$

Term T_4 can be bounded by hiding in the penalty term and term T_5 can be bounded by using inverse inequalities,

$$\begin{aligned}|T_4| &\le \epsilon\frac{r_{\text{transp}}^2}{Kh}\sum_{\gamma\in\Gamma_h}\left\|\left[\xi^A\right]\right\|_{0,\gamma}^2+\frac{Kh}{r_{\text{transp}}^2}\sum_{E\in\mathcal{E}_h}\left\|\nabla\xi^I\cdot\mathbf{n}\right\|_{0,\partial E}^2\\ &\le \epsilon J_{0,\text{transp}}\left(\xi^A,\xi^A\right)+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-1}}\|c\|_s^2,\end{aligned}$$

$$\begin{aligned}
|T_5| &\leq \frac{\epsilon h}{K r_{\text{transp}}^2} \sum_{E\in\mathcal{E}_h} \left\| \mathbf{D}^{\frac{1}{2}}(\mathbf{u}) \nabla \xi^A \cdot \mathbf{n} \right\|_{0,\partial E}^2 + \frac{K r_{\text{transp}}^2}{h} \sum_{E\in\mathcal{E}_h} \left\| \xi^I \right\|_{0,\partial E}^2 \\
&\leq \epsilon \| \mathbf{D}^{\frac{1}{2}}(\mathbf{u}) \nabla \xi^A \|_0^2 + K \frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}} \| c \|_s^2 .
\end{aligned}$$

The terms T_6 through T_8 can be bounded by using Cauchy-Schwartz inequality and approximation results,

$$\begin{aligned}
|T_6| &\leq \epsilon J_{0,\text{transp}}\left(\xi^A, \xi^A\right) + K \frac{h}{r_{\text{transp}}^2} \frac{h^{2\mu_{\text{transp}}-1}}{r_{\text{transp}}^{2s_{\text{transp}}-1}} \| c \|_s^2 \\
&\leq \epsilon J_{0,\text{transp}}\left(\xi^A, \xi^A\right) + K \frac{h^{2\mu_{\text{transp}}}}{r_{\text{transp}}^{2s_{\text{transp}}+1}} \| c \|_s^2 , \\
|T_7| &\leq K \frac{h}{r_{\text{transp}}^2} \sum_{\gamma\in\Gamma_{h,\text{out}}} \int_\gamma \left(\xi^A\right)^2 + K \frac{r_{\text{transp}}^2}{h} \frac{h^{2\mu_{\text{transp}}-1}}{r_{\text{transp}}^{2s_{\text{transp}}-1}} \| c \|_s^2 \\
&\leq K \left\| \sqrt{\phi} \xi^A \right\|_0^2 + K \frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}} \| c \|_s^2 , \\
|T_8| &\leq \epsilon J_{0,\text{transp}}\left(\xi^A, \xi^A\right) + K J_0^\sigma\left(\xi^I, \xi^I\right) \\
&\leq \epsilon J_{0,\text{transp}}\left(\xi^A, \xi^A\right) + K \frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}} \| c \|_s^2 .
\end{aligned}$$

We remark that, in the cases of triangles or tetrahedra, we can choose a continuous $\widehat{c}$ to have $T_5 = T_8 = 0$, but this does not help unless we have a sharper bound on T_7.

Noting that $[c] = 0$, and $\mathbf{u}^M \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}$ on the domain boundary if the constant M for the cut-off operator is sufficiently large, we can write the last two terms in the right hand side of of error equation (5.7) as

$$\begin{aligned}
& B\left(c, \xi^A; \mathbf{u}\right) - B\left(c, \xi^A; \mathbf{u}^M\right) \\
= & \sum_{E\in\mathcal{E}_h} \int_E \left(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}^M)\right) \nabla c \cdot \nabla \xi^A - \sum_{E\in\mathcal{E}_h} \int_E c\left(\mathbf{u} - \mathbf{u}^M\right) \cdot \nabla \xi^A \\
& - \sum_{\gamma\in\Gamma_h} \int_\gamma \left\{ \left(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}^M)\right) \nabla c \cdot \mathbf{n}_\gamma \right\} \left[\xi^A\right] \\
& - s_{\text{transp}} \sum_{\gamma\in\Gamma_h} \int_\gamma \left\{ \left(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}^M)\right) \nabla \xi^A \cdot \mathbf{n}_\gamma \right\} [c] \\
& + \sum_{\gamma\in\Gamma_h} \int_\gamma c^* \left(\mathbf{u} - \mathbf{u}^M\right) \cdot \mathbf{n}_\gamma \left[\xi^A\right] \\
=: & \sum_{i=1}^5 S_i .
\end{aligned}$$

Noting that $\left|\mathbf{u}-\mathbf{u}^M\right| \leq \left|\mathbf{u}-\mathbf{u}^{DG}\right|$ point-wise if the constant M for the cut-off operator is sufficiently large, we can bound term S_1 as

$$\begin{aligned}
|S_1| &\leq K \sum_{E\in\mathcal{E}_h} \int_E \left|\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u}^M)\right| \left|\nabla\xi^A\right| \\
&\leq \epsilon \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2 + K\left\|\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u}^M)\right\|_0^2 \\
&\leq \epsilon \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2 + K\left\|\mathbf{u}-\mathbf{u}^M\right\|_0^2 \\
&\leq \epsilon \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2 + K\left\|\mathbf{u}-\mathbf{u}^{DG}\right\|_0^2 .
\end{aligned}$$

Term S_2 can be bounded in a similar way as that for S_1:

$$\begin{aligned}
|S_2| &\leq K \sum_{E\in\mathcal{E}_h} \int_E \left|\mathbf{u}-\mathbf{u}^M\right| \left|\nabla\xi^A\right| \\
&\leq \epsilon \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2 + K\left\|\mathbf{u}-\mathbf{u}^{DG}\right\|_0^2 .
\end{aligned}$$

Term S_3 can be bounded by using the continuity of dispersion/diffusion tensor and the penalty term.

$$\begin{aligned}
|S_3| &\leq K \sum_{\gamma\in\Gamma_h} \int_\gamma \left|\left\{\mathbf{D}(\mathbf{u})-\mathbf{D}(\mathbf{u}^M)\right\}\right| \left|\left[\xi^A\right]\right| \\
&\leq K \sum_{\gamma\in\Gamma_h} \left\|\left[\xi^A\right]\right\|_{0,\gamma} \\
&\quad \cdot \sum_{E_i\cap E_j=\gamma\in\Gamma_h} \left(\left\| \mathbf{D}(\mathbf{u})|_{E_i} - \mathbf{D}(\mathbf{u}^M)|_{E_i} \right\|_{0,\gamma} + \left\| \mathbf{D}(\mathbf{u})|_{E_j} - \mathbf{D}(\mathbf{u}^M)|_{E_j} \right\|_{0,\gamma} \right) \\
&\leq K \sum_{E_i\cap E_j=\gamma\in\Gamma_h} \left(\left\| \mathbf{u}^{DG}|_{E_i} - \mathbf{u} \right\|_{0,\gamma} + \left\| \mathbf{u}^{DG}|_{E_j} - \mathbf{u} \right\|_{0,\gamma} \right) \sum_{\gamma\in\Gamma_h} \left\|\left[\xi^A\right]\right\|_{0,\gamma} \\
&\leq K \frac{h}{r_{\text{transp}}^2} \sum_{E_i\cap E_j=\gamma\in\Gamma_h} \left(\left\| \mathbf{u}^{DG}|_{E_i} - \mathbf{u} \right\|_{0,\gamma}^2 + \left\| \mathbf{u}^{DG}|_{E_j} - \mathbf{u} \right\|_{0,\gamma}^2 \right) \\
&\quad + \epsilon J_{0,\text{transp}}\left(\xi^A,\xi^A\right) .
\end{aligned}$$

Term S_4 vanishes because $[c]=0$. Term S_5 can be bounded using penalty terms and the error of normal velocity on element interfaces.

$$\begin{aligned}
|S_5| &\leq K \sum_{\gamma\in\Gamma_h} \int_\gamma \left|\mathbf{u}\cdot\mathbf{n}_\gamma - \mathbf{u}^M\cdot\mathbf{n}_\gamma\right| \left|\left[\xi^A\right]\right| \\
&\leq K \frac{h}{r_{\text{transp}}^2} \sum_{\gamma\in\Gamma_h} \int_\gamma \left|\mathbf{u}\cdot\mathbf{n}_\gamma - \mathbf{u}^{DG}\cdot\mathbf{n}_\gamma\right|^2 + \epsilon J_{0,\text{transp}}\left(\xi^A,\xi^A\right) .
\end{aligned}$$

Combining all the terms in (5.7), we have,

$$\begin{aligned}
&\frac{d}{dt}\left\|\sqrt{\phi}\xi^A\right\|_0^2 + \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2 + J_0^\sigma\left(\xi^A,\xi^A\right) \\
\leq\ & K\left\|\sqrt{\phi}\xi^A\right\|_0^2 + K\left\|\mathbf{u}-\mathbf{u}^{DG}\right\|_0^2 + K\frac{h}{r_{\text{transp}}^2} \sum_{\gamma\in\Gamma_h} \left\|\mathbf{u}\cdot\mathbf{n}_\gamma - \mathbf{u}^{DG}\cdot\mathbf{n}_\gamma\right\|_{0,\gamma}^2 \\
&+K\frac{h}{r_{\text{transp}}^2} \sum_{E_i\cap E_j=\gamma\in\Gamma_h} \left(\left\| \mathbf{u}^{DG}|_{E_i} - \mathbf{u} \right\|_{0,\gamma}^2 + \left\| \mathbf{u}^{DG}|_{E_j} - \mathbf{u} \right\|_{0,\gamma}^2 \right)
\end{aligned}$$

$$+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}}.$$

Integrating with respect to time, we have,

$$\begin{aligned}
&\left\|\sqrt{\phi}\xi^A\right\|_0^2(t)+\int_0^t \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\xi^A\|_0^2(\tau)d\tau+\int_0^t J_0^{\sigma}\left(\xi^A,\xi^A\right)(\tau)d\tau\\
\leq\;&\left\|\sqrt{\phi}\xi^A\right\|_0^2(0)+K\int_0^t\left\|\sqrt{\phi}\xi^A\right\|_0^2(\tau)d\tau+K\int_0^t\left\|\mathbf{u}-\mathbf{u}^{DG}\right\|_0^2(\tau)d\tau\\
&+K\frac{h}{r_{\text{transp}}^2}\int_0^t\sum_{E_i\cap E_j=\gamma\in\Gamma_h}\left(\left\|\mathbf{u}^{DG}\big|_{E_i}-\mathbf{u}\right\|_{0,\gamma}^2+\left\|\mathbf{u}^{DG}\big|_{E_j}-\mathbf{u}\right\|_{0,\gamma}^2\right)(\tau)d\tau\\
&+K\frac{h}{r_{\text{transp}}^2}\int_0^t\sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}\cdot\mathbf{n}_\gamma-\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma\right\|_{0,\gamma}^2(\tau)d\tau+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}}\\
\leq\;&K\int_0^t\left\|\sqrt{\phi}\xi^A\right\|_0^2(\tau)d\tau+K\int_0^t\left\|\mathbf{u}-\mathbf{u}^{DG}\right\|_0^2(\tau)d\tau\\
&+K\frac{h}{r_{\text{transp}}^2}\int_0^t\sum_{E_i\cap E_j=\gamma\in\Gamma_h}\left(\left\|\mathbf{u}^{DG}\big|_{E_i}-\mathbf{u}\right\|_{0,\gamma}^2+\left\|\mathbf{u}^{DG}\big|_{E_j}-\mathbf{u}\right\|_{0,\gamma}^2\right)(\tau)d\tau\\
&+K\frac{h}{r_{\text{transp}}^2}\int_0^t\sum_{\gamma\in\Gamma_h}\left\|\mathbf{u}\cdot\mathbf{n}_\gamma-\mathbf{u}^{DG}\cdot\mathbf{n}_\gamma\right\|_{0,\gamma}^2(\tau)d\tau+K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}}.
\end{aligned}$$

The theorem follows from approximation results and the triangle inequality. □

6. The coupled system of flow and transport

Combining the results developed for the flow and for the transport equations, we can obtain error estimates for the coupled system.

Theorem 6.1. (Error estimate for transport in the coupled system) *Let $(\mathbf{u},p,c)$ be the solution to (2.1)-(2.7), and assume $p\in L^2\left(0,T;H^{s_{\text{flow}}}(\mathcal{E}_h)\right)$, $\mathbf{u}\in\left(L^2\left(0,T;H^{s_{\text{flow}}-1}(\mathcal{E}_h)\right)\right)^d$, $c\in L^2\left(0,T;H^{s_{\text{transp}}}(\mathcal{E}_h)\right)$, $\partial c/\partial t\in L^2\left(0,T;H^{s_{\text{transp}}-1}(\mathcal{E}_h)\right)$ and $c_0\in H^{s_{\text{transp}}-1}(\mathcal{E}_h)$. We further assume that p, ∇p, c and ∇c are essentially bounded, and r_{flow} is in the same order with r_{transp} (i.e. $r_{\text{flow}}/r_{\text{transp}}$ and $r_{\text{transp}}/r_{\text{flow}}$ are bounded). If the constant M for the cut-off operator and the penalty parameters are sufficiently large, then there exists a constant K independent of h and r such that*

$$\begin{aligned}
&\left\|\sqrt{\phi}\left(C^{DG}-c\right)\right\|_{L^\infty(0,T;L^2)}+\|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\left(C^{DG}-c\right)\|_{L^2(0,T;L^2)}\\
&\left(\int_0^T J_{0,\text{transp}}\left(C^{DG}-c,C^{DG}-c\right)(\tau)d\tau\right)^{1/2}\\
\leq\;&K\frac{h^{\mu_{\text{flow}}-1}}{r_{\text{flow}}^{s_{\text{flow}}-1-\delta/2}}+K\frac{h^{\mu_{\text{transp}}-1}}{r_{\text{transp}}^{s_{\text{transp}}-3/2}},
\end{aligned}\tag{6.1}$$

where $\mu_{\text{flow}} = \min(r_{\text{flow}}+1, s_{\text{flow}})$, $\mu_{\text{transp}} = \min(r_{\text{transp}}+1, s_{\text{transp}})$, $r_{\text{flow}} \geq 1$, $s_{\text{flow}} \geq 2$, $r_{\text{transp}} \geq 1$, $s_{\text{transp}} \geq 2$, *and* $\delta = 0$ *in the cases of conforming meshes with triangles or tetrahedra. In general cases,* $\delta = 1$.

Proof. Substituting the DG velocity error bounds (4.6) and (4.7) into the error estimate for concentration (5.4), we have

$$\begin{aligned}
& \left\| \sqrt{\phi}\left(C^{DG}-c\right)\right\|_0^2(t) + \int_0^t \|\mathbf{D}^{\frac{1}{2}}(\mathbf{u})\nabla\left(C^{DG}-c\right)\|_0^2(\tau)d\tau \\
& + \int_0^t J_0^\sigma\left(C^{DG}-c, C^{DG}-c\right)(\tau)d\tau \\
\leq\; & K\int_0^t \left\|\sqrt{\phi}\left(C^{DG}-c\right)\right\|_0^2(\tau)d\tau + K\left(1+\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}\right)\int_0^t \left\|C^{DG}-c\right\|_0^2(\tau)d\tau \\
& + K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}} + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}} + K\frac{h}{r_{\text{transp}}^2} \\
& \cdot\left(\int_0^t\left(\frac{r_{\text{flow}}^2}{h}+\frac{r_{\text{transp}}^2}{h}\right)\left\|c-C^{DG}\right\|_0^2(\tau)d\tau + \frac{h^{2\mu_{\text{transp}}-1}}{r_{\text{transp}}^{2s_{\text{transp}}-2}} + \frac{h^{2\mu_{\text{flow}}-3}}{r_{\text{flow}}^{2s_{\text{flow}}-4-\delta}}\right) \\
& + K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}} \\
\leq\; & K\left(1+\frac{r_{\text{transp}}^2}{r_{\text{flow}}^2}+\frac{r_{\text{flow}}^2}{r_{\text{transp}}^2}\right)\int_0^t\left\|\sqrt{\phi}\left(C^{DG}-c\right)\right\|_0^2(\tau)d\tau + K\frac{h^{2\mu_{\text{transp}}}}{r_{\text{flow}}^2 r_{\text{transp}}^{2s_{\text{transp}}-2}} \\
& + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}} + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{transp}}^2 r_{\text{flow}}^{2s_{\text{flow}}-4-\delta}} + K\frac{h^{2\mu_{\text{transp}}-2}}{r_{\text{transp}}^{2s_{\text{transp}}-3}}.
\end{aligned}$$

Using the Gronwall's inequality, we have the error bound (6.1) for concentration. □

Remark 6.2. Let us assume that the exact solutions p, $\mathbf{u}$, c are sufficiently smooth and $r_{\text{transp}} = r_{\text{flow}}$. Theorem 6.1 gives sharp *a priori* error estimates in the following senses. The error bound in $L^2\left(H^1\right)$ for concentration is optimal in h and nearly optimal in p with a loss of power $\frac{1}{2}$. The $L^2(L^2)$ error estimate for concentration jump is optimal in h and in p.

Theorem 6.3. (Error estimate for flow part in the coupled system) *Let* $(\mathbf{u}, p, c)$ *be the solution to (2.1)-(2.7), and assume* $p \in L^2\left(0,T;H^{s_{\text{flow}}}(\mathcal{E}_h)\right)$, $\mathbf{u} \in \left(L^2\left(0,T;H^{s_{\text{flow}}-1}(\mathcal{E}_h)\right)\right)^d$, $c \in L^2\left(0,T;H^{s_{\text{transp}}}(\mathcal{E}_h)\right)$, $\partial c/\partial t \in L^2\left(0,T;H^{s_{\text{transp}}-1}(\mathcal{E}_h)\right)$ *and* $c_0 \in H^{s_{\text{transp}}-1}(\mathcal{E}_h)$. *We further assume that* p, ∇p, c *and* ∇c *are essentially bounded, and* r_{flow} *is in the same order with* r_{transp} *(i.e.* $r_{\text{flow}}/r_{\text{transp}}$ *and* $r_{\text{transp}}/r_{\text{flow}}$ *are bounded). If the constant* M *for the cut-off operator and the penalty parameters are sufficiently large, then there exists a constant* K *independent of* h *and* r *such that*

$$\left\|\mathbf{K}^{1/2}(c)\nabla\left(P^{DG}-p\right)\right\|_{L^\infty(0,T;L^2)} \tag{6.2}$$

$$\leq \quad K\frac{h^{\mu_{\text{flow}}-1}}{r_{\text{flow}}^{s_{\text{flow}}-1-\delta/2}} + K\frac{h^{\mu_{\text{transp}}-1}}{r_{\text{transp}}^{s_{\text{transp}}-3/2}},$$

(6.3)
$$\sup_{t\in(0,T)} \left(J_{0,\text{flow}}\left(P^{DG}-p, P^{DG}-p\right)\right)^{1/2}$$
$$\leq \quad K\frac{h^{\mu_{\text{flow}}-1}}{r_{\text{flow}}^{s_{\text{flow}}-3/2}} + K\frac{h^{\mu_{\text{transp}}-1}}{r_{\text{transp}}^{s_{\text{transp}}-3/2}}$$

and

(6.4)
$$\left\|\mathbf{u}^{DG}-\mathbf{u}\right\|_{L^\infty(0,T;L^2)}$$
$$\leq \quad K\frac{h^{\mu_{\text{flow}}-1}}{r_{\text{flow}}^{s_{\text{flow}}-1-\delta/2}} + K\frac{h^{\mu_{\text{transp}}-1}}{r_{\text{transp}}^{s_{\text{transp}}-3/2}},$$

where $\mu_{\text{flow}} = \min\left(r_{\text{flow}}+1, s_{\text{flow}}\right)$*,* $\mu_{\text{transp}} = \min\left(r_{\text{transp}}+1, s_{\text{transp}}\right)$*,* $r_{\text{flow}} \geq 1$*,* $s_{\text{flow}} \geq 2$*,* $r_{\text{transp}} \geq 1$*,* $s_{\text{transp}} \geq 2$*, and* $\delta = 0$ *in the cases of conforming meshes with triangles or tetrahedra. In general cases,* $\delta = 1$*.*

Proof. Taking L^∞ norm with time in (4.1), we have

$$\left\|\mathbf{K}^{1/2}(c)\nabla\left(P^{DG}-p\right)\right\|^2_{L^\infty(0,T;L^2)}$$
$$\leq \quad K\left(1+\frac{r^2_{\text{transp}}}{r^2_{\text{flow}}}\right)\left\|c-C^{DG}\right\|^2_{L^\infty(0,T;L^2)} + K\frac{h^{2\mu_{\text{transp}}}}{r^2_{\text{flow}}r_{\text{transp}}^{2s_{\text{transp}}-2}} + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}}.$$

Substituting (6.1) into above inequality, we obtain (6.2). Similarly, L^∞ norm of (4.2) gives

$$\sup_{t\in(0,T)} J_{0,\text{flow}}\left(P^{DG}-p, P^{DG}-p\right)$$
$$\leq \quad K\left(1+\frac{r^2_{\text{transp}}}{r^2_{\text{flow}}}\right)\left\|c-C^{DG}\right\|^2_{L^\infty(0,T;L^2)} + K\frac{h^{2\mu_{\text{transp}}}}{r^2_{\text{flow}}r_{\text{transp}}^{2s_{\text{transp}}-2}} + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-3}}.$$

Substituting (6.1) into above inequality, we obtain (6.3). L^∞ norm of (4.6) gives

$$\left\|\mathbf{u}^{DG}-\mathbf{u}\right\|^2_{L^\infty(0,T;L^2)}$$
$$\leq \quad K\left(1+\frac{r^2_{\text{transp}}}{r^2_{\text{flow}}}\right)\left\|c-C^{DG}\right\|^2_{L^\infty(0,T;L^2)} + K\frac{h^{2\mu_{\text{transp}}}}{r^2_{\text{flow}}r_{\text{transp}}^{2s_{\text{transp}}-2}} + K\frac{h^{2\mu_{\text{flow}}-2}}{r_{\text{flow}}^{2s_{\text{flow}}-2-\delta}}.$$

Substituting (6.1) into above inequality, we obtain (6.4). □

Remark 6.4. Let us assume that the exact solutions p, $\mathbf{u}$, c are sufficiently smooth and $r_{\text{transp}} = r_{\text{flow}}$. Theorem 6.3 gives sharp *a priori* error estimates in the following senses. The semi-$L^2\left(H^1\right)$ error bound for pressure is optimal in h and nearly optimal in p with a loss of power $\frac{1}{2}$. The $L^\infty(L^2)$ convergence for pressure jump is optimal in h and in p. Finally, the $L^\infty\left(L^2\right)$ error estimate for velocity establishes optimality in h and sub-optimality in p by $\frac{1}{2}$.

7. Discussion and conclusion

Three versions of primal discontinuous Galerkin (DG) methods were proposed to solve coupled flow and reactive transport in porous media. A cut-off operator $\mathcal{M}$ was introduced in the DG schemes in order to achieve convergence. We estimated the uniform positive definitiveness and the uniform Lipschitz continuity of the well known established dispersion/diffusion tensor for porous media.

We first studied the flow and the transport parts separately assuming the error from the other part is known. Then we combined the results from the two parts to complete the *a priori* error analysis of the coupled system. We consider the same mesh for both flow and transport, but the transport part can use a polynomial degree of approximation r_{transp} different from that for the flow part r_{flow}. A set of conditions were proposed for convergence of DG applied to the coupled system. Interestingly, the polynomial degrees of approximation spaces for flow and for transport needs to be in the same order in order to maintain the convergence of DG applied to the coupled system. That is, $r_{\text{flow}}/r_{\text{transp}}$ and $r_{\text{transp}}/r_{\text{flow}}$ need to be bounded. This excludes unbalance p-version refinement for flow and for transport. For example, p-version of DG with $r_{\text{flow}} = r_{\text{transp}}^2$ for the coupled system might not converge.

If the degree of approximation for the flow part differs from that for the transport part, the convergence behaviors for the coupled system are controlled by the part with less degree of approximation. Optimal or nearly optimal convergences for both flow and transport can be achieved when the same polynomial degrees of approximation spaces were chosen for flow and transport. Under this condition, the error bound in $L^2\left(H^1\right)$ for concentration is optimal in h and nearly optimal in p with a loss of power $\frac{1}{2}$. The semi-$L^2\left(H^1\right)$ error bound for pressure is optimal in h and nearly optimal in p with a loss of power $\frac{1}{2}$. The $L^\infty\left(L^2\right)$ error estimate for velocity establishes optimality in h and sub-optimality in p by $\frac{1}{2}$. Finally, the $L^2(L^2)$ error estimate for concentration jump and the $L^\infty(L^2)$ estimate for pressure jump are optimal in h and in p.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, 1975.

[2] T. Arbogast, S. Bryant, C. Dawson, F. Saaf, C. Wang, and M. Wheeler. Computational methods for multiphase flow and reactive transport problems arising in subsurface contaminant remediation. *J. Comput. Appl. Math.*, 74(1-2):19–32, 1996.

[3] T. Arbogast and M. F. Wheeler. A characteristics-mixed finite element method for advection-dominated transport problems. *SIAM J. Numer. Anal.*, 32:404–424, 1995.

[4] D. N. Arnold. *An Interior Penalty Finite Element Method with Discontinuous Elements.* PhD thesis, The University of Chicage, Chicago, IL, 1979.

[5] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM Journal on Numerical Analysis*, 19:742–760, 1982.

[6] I. Babuska and M. Suri. The h-p version of the finite element method with quasi-uniform meshes. *RAIRO Model. Math Anal. Numer.*, 21:199–238, 1987.

[7] I. Babuska and M. Suri. The optimal convergence rates of the p version of the finite element methos. *SIAM J. Numer. Anal.*, 24:750–776, 1987.

[8] C. E. Baumann and J. T. Oden. A discontinuous *hp* finite element method for convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.*, 175(3-4):311–341, 1999.

[9] C. M. Bethke, J. van der Lee, and J. M. Schmitt. The chemistry beneath our feet. modeling reacting flow in the earth's crust. In *Scientific bridges for 2000 and beyond*. Académie des Sciences, 1999.

[10] R. C. Borden and P. B. Bedient. Transport of dissolved hydrocarbons influenced by oxygen-limited biodegradation 1. theoretical development. *Water Resources Research*, 22:1973–1982, 1986.

[11] S. L. Bryant and K. E. Thompson. Theory, modeling and experiment in reactive transport in porous media. *Current Opinion in Colloid and Interface Science*, 6:217–222, 2001.

[12] C. Y. Chiang, C. N. Dawson, and M. F. Wheeler. Modeling of in-situ biorestoration of organic compounds in groundwater. *Transport in Porous Media*, 6:667–702, 1991.

[13] C. Dawson, S. Sun, and M. F. Wheeler. Compatible algorithms for coupled flow and transport. *Comput. Meth. Appl. Mech. Eng.* to appear.

[14] C. N. Dawson. Godunov-mixed methods for advection-diffusion equations in multidimensions. *SIAM J. Numer. Anal.*, 30:1315–1332, 1993.

[15] C. N. Dawson, T. F. Russell, and M. F. Wheeler. Some improved error estimates for the modified method of characteristics. *SIAM J. Numer. Anal.*, 26:1487–1512, 1989.

[16] C. N. Dawson and M. F. Wheeler. Time-splitting methods for advection-diffusion-reaction equations arising in contaminant transport. In Jr. R. E. O'Malley, editor, *Proceedings of the second International Conference on Industrial and Applied Mathematics, ICIAM'91*, pages 71–82, Philadelphia, 1991. SIAM.

[17] P. Engesgaard and K. L. Kipp. A geochemical transport model for redox-controlled movement of mineral fronts in groundwater flow systems: A case of nitrate removal by oxidation of pyrite. *Water Resources Research*, 28(10):2829–2843, 1992.

[18] A. K. Gupta, P. R. Bishnoi, and N. Kalogerakis. A method for the simultaneous phase equilibria and stability calculations for multiphase reacting and non-reacting systmes. *Fluid Phase Equilibria*, 63(8):65–89, 1991.

[19] J. F. Kanney, C. T. Miller, and D. A. Barry. Comparison of fully coupled approaches for approximating nonlinear transport and reaction problems. *in review*, 2002.

[20] J. S. Kindred and M. A. Celia. Contaminant transport and biodegradation 2. conceptual model and test simulations. *Water Resources Research*, 25:1149–1159, 1989.

[21] F. M. Morel and J. G. Hering. *Principles and Applications of Aquatic Chemistry.* John Wiley and Sons, 1993.

[22] J. T. Oden, I. Babuška, and C. E. Baumann. A discontinuous hp finite element method for diffusion problems. *J. Comput. Phys.*, 146:491–516, 1998.

[23] B. Rivière. *Discontinuous Galerkin finite element methods for solving the miscible displacement problem in porous media*. PhD thesis, The University of Texas at Austin, 2000.

[24] B. Rivière and M. F. Wheeler. Non conforming methods for transport with nonlinear reaction. *Contemporary Mathematics*, 295:421–432, 2002.

[25] B. Rivière, M. F. Wheeler, and V. Girault. Part I: Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. *Computational Geosciences*, 3:337–360, 1999.

[26] B. Rivière, M. F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM J. Numer. Anal.*, 39(3):902–931, 2001.

[27] J. Rubin. Transport of reacting solutes in porous media: Relation between mathematical nature of problem formulation and chemical nature of reactions. *Water Resources Research*, 19(5):1231–1252, 1983.

[28] J. Rubin and R. V. James. Dispersion-affected transport of reacting solutes in saturated porous media: Galerkin method applied to equilibrium-controlled exchange in unidirectional steady water flow. *Water Resources Research*, 9(5):1332–1356, 1973.

[29] F. Saaf. *A study of reactive transport phenomena in porous media*. PhD thesis, Rice University, 1996.

[30] Ch. Schwab. *p- and hp- finite element methods, theory and applications in solid and fluid mechanics*. Oxford science publications, 1998.

[31] J. V. Smith, R. W. Missen, and W. R. Smith. General optimality criteria for multiphase multireaction chemcial equilibrium. *AIChE Journal*, 39(4):707–710, 1993.

[32] C. I. Steefel and P. Van Cappellen. Special issue: Reactive transport modeling of natural systems. *Journal of Hydrology*, 209(1-4):1–388, 1998.

[33] S. Sun. *Discontinuous Galerkin methods for reactive transport in porous media*. PhD thesis, The University of Texas at Austin, 2003.

[34] S. Sun, B. Rivière, and M. F. Wheeler. A combined mixed finite element and discontinuous Galerkin method for miscible displacement problem in porous media. In *Recent progress in computational and applied PDEs, conference proceedings for the international conference held in Zhangjiaje in July 2001*, 319-346.

[35] S. Sun and M. F. Wheeler. Energy norm *a posteriori* error estimation for discontinuous Galerkin approximations of reactive transport problems. TICAM report 03-39, Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX, 2003.

[36] S. Sun and M. F. Wheeler. *A posteriori* error analyses for symmetric discontinuous Galerkin approximations of reactive transport problems. TICAM report 03-19, Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX, 2003.

[37] A. J. Valocchi and M. Malmstead. Accuracy of operator splitting for advection-dispersion-reaction problems. *Water Resources Research*, 28(5):1471–1476, 1992.

[38] J. van der Lee and L. De Windt. Present state and future directions of modeling of geochemistry in hydrogeological systems. *J. Contam. Hydrol.*, 47/2(4):265–282, 2000.

[39] M. Th. van Genuchten. Analytical soultions for chemical transport with simultaneous adsorption, zero-order production and first-order decay. *Journal of Hydrology*, 49:213–233, 1981.

[40] M. F. Wheeler and B. L. Darlow. Interior penalty Galerkin procedures for miscible displacement problems in porous media. In *Computational methods in nonlinear mechanics (Proc. Second Internat. Conf., Univ. Texas, Austin, Tex., 1979)*, pages 485–506. North-Holland, Amsterdam, 1980.

[41] M. F. Wheeler and C. N. Dawson. An operator-splitting method for advection-diffusion-reaction problems. In J. R. Whiteman, editor, *The Mathematics of Finite Elements and Applications VI*, pages 463–382. Academic Press, London, 1987.

[42] M. F. Wheeler, C. N. Dawson, P. B. Bedient, C. Y. Chiang, R. C. Bordern, and H. S. Rifai. Numerical simulation of microbial biodegradation of hydrocarbons in groundwater. In *Proceedings of AGWSE/IGWMCH Conference on Solving Ground Water Problems with Models, National Water Wells Association*, pages 92–108, 1987.

[43] M. F. Wheeler, O. Eslinger, S. Sun, and B. Rivière. Discontinuous Galerkin method for modeling flow and reactive transport in porous media. In *Proceedings of the 2002 CANUM conference, series ESAIM.* to appear.

[44] M.F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15(1):152–161, 1978.

[45] G. T. Yeh and V. S. Tripathi. A critical evaluation of recent developments in hydrogeochemical transport models of reactive multichemical components. *Water Resources Research*, 25(1):93–108, 1989.

[46] G. T. Yeh and V. S. Tripathi. A model for simulating transport of reactive multispecies components: model development and demonstration. *Water Resources Research*, 27(12):3075–3094, 1991.