POLYNOMIAL EXACT SEQUENCES AND PROJECTION-BASED INTERPOLATION WITH APPLICATION TO MAXWELL EQUATIONS

L. Demkowicz

Institute for Computational Engineering and Sciences
The University of Texas at Austin

Abstract

We review the construction of polynomial exact sequences, the theory of projection based interpolation, and discuss their application to the convergence analysis for conforming hp-Finite Element discretization of time-harmonic Maxwell equations.

Key words: Maxwell's equations, hp finite elements

AMS subject classification: 65N30, 35L15

Acknowledgment

The work has been supported by Air Force under Contract F49620-98-1-0255. The work was done in collaboration and with help of many friends: Analisa Buffa, Weiming Cao, Mark Ainsworth, Ivo Babuška, Jay Gopalakrishnan, Peter Monk, Waldek Rachowicz, and Joachim Schoeberl. The author would also like to thank Patrick Rabier for discussions on the subject.

1 Introduction

The presented notes review the concept and main results concerning commuting projections and projection-based interpolation operators defined for one-, two- and three-dimensional exact sequences involving the gradient, curl and divergence operators, and Sobolev spaces. The discrete sequences correspond to polynomial spaces defining the classical, continuous finite elements, the "edge elements" of Nédélec, and "face elements" of Raviart-Thomas. All discussed results extend to the elements of variable order as well as parametric elements. The presentation reproduces results for 2D from [18] and 3D from [16, 22, 19, 12] and attempts to present them in a unified manner for all types of finite elements forming the exact sequences. The idea of the projection-based interpolation for elliptic problems was introduced in [33] and generalized to the exact sequence in [20]. The presented results hinge on the existence of polynomial preserving extension operators (a work still under completion [36]).

Sobolev spaces. I am assuming that the reader is familiar with essentials of Sobolev spaces. For those who seek a complete and compact presentation on the subject, I highly recommend the book of McLean [29] to which I will refer for most of technical details relevant to this paper. We will use the Hörmander's definition for spaces $H^s(\Omega)$ that remains valid for the whole range of $s \in \mathbb{R}$. The H^s spaces are isomorphic with duals of spaces $\widetilde{H}^s$, the closure of $\mathcal{D}(\Omega)$ in $H^s(\mathbb{R}^n)$. For $s \geq -\frac{1}{2}$, the restriction operator from $\widetilde{H}^s$ into $H^s(\Omega)$ is injective and, for this range of s, space $\widetilde{H}^s$ can be identified with a subspace of $H^s(\Omega)$. For values s different from half-integers, space $\widetilde{H}^s(\Omega)$ coincides with the space $H^s_0(\Omega)$, the closure of test functions in the $H^s(\Omega)$-norm, with the equivalence constants blowing up with s approaching the half-integers. The energy spaces $H^1(\Omega), \boldsymbol{H}(\mathbf{curl}, \Omega), \boldsymbol{H}(\mathrm{div}, \Omega)$ are imposed by physics, and so are the corresponding spaces of boundary traces: $H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega), \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)$. The non-locality of norms and the break-down of Trace Theorem and so-called localization results for the half-integers, are the source of notorious technical difficulties. The use of fractional spaces H^s and a careful monitoring of equivalence constants allows for alleviating most of these difficulties as we shall present it in the text.

Throughout the paper, Ω will denote a single element, interval in 1D, a polygon in 2D, or a polyhedron in 3D. The domains fall into the general category of Lipshitz domains covered by McLean for scalar-valued functions. For details concerning the vector-valued spaces, we refer to the work of Buffa and Ciarlet [11].

We use the higher-order Sobolev spaces to express regularity of projected and interpolated functions. It has been well established that this is a wrong choice for elliptic of Maxwell problems formulated in polyhedral domains or/and material interfaces. Most of the research on hp methods and exponential convergence is based on the notion of countably normed Besov spaces introduced by Babuška and Guo, see e.g. [37]. It is for that reason that we always try to estimate the interpolation errors with the corresponding best approximation errors. The last step of the interpolation error estimation resulting in optimal p- or hp-convergence rates reduces then to the best approximation results using more sophisticated means to access the regularity of approximated functions.

Scope of the presentation. The following four chapters correspond to four lectures. In the first lecture we discuss the grad-curl-div exact sequence and review the known polynomial exact sequences corresponding to various finite elements, and the concept of parametric elements. This part is mostly algebraic, although some of the details and even the notation may be a little overwhelming for a first time reader of the material. The second lecture focuses on a seemingly trivial one-dimensional sequence. We proceed with an attention to details and invoke already at this level the main arguments and details on Sobolev spaces. The third lecture covers the two-dimensional case. Finally the fourth lecture proceeds at a faster pace zooming through the three-dimensional case covered in Section 5, and discussing applications of the presented techniques to the analysis and approximation of time-harmonic Maxwell equations. We conclude with a short discussion of open problems. Contrary to the original contribution [19], this presentation "marches" from 1D to 3D problems.

2 Exact Polynomial Sequences

2.1 One-dimensional sequences

We begin our discussion with the simplest one-dimensional exact sequence.

$$\mathbb{R} \rightarrow H^s(I) \xrightarrow{\partial} H^{s-1}(I) \rightarrow \{0\} \tag{2.1}$$

Here $s \geq 0$, both Sobolev spaces are defined on the unit interval $I = (0,1)$ and ∂ denotes the derivative operator. The space of real numbers $\mathbb{R}$ symbolizes the one-dimensional space of constant functions, and $\{0\}$ denotes the trivial space consisting of the zero function only. The first operator (not shown) is identity, and the last one is the trivial map setting all arguments to the zero vector. The notion of the exact sequence conveys in this case the non-so-trivial (in context of real s) message that the derivative operator is well-defined, it is a surjection, and that its null space consists of constants. Let H^s_{avg} denote the subspace of functions of zero average,

$$H^s_{avg}(I) = \{u \in H^s \ : \ \int_I u = 0\} \ . \tag{2.2}$$

The exact sequence property is a consequence of the following result.

Proposition 1

The derivative operator is an isomorphism from $H^s_{avg}(I)$ onto $H^{s-1}(I)$. ∎

Proof:

- We first demonstrate that ∂ is well-defined. Recall first [29, p.309] that there exists a continuous extension operator,

  $$H^s(I) \ni u \rightarrow U \in H^s(\mathbb{R}) \ . \tag{2.3}$$

  Take and arbitrary $\phi \in \mathcal{D}(I)$. We have,

  $$\begin{aligned} | < u', \phi > | \quad &= |- < U, \phi' > | \\ &\leq \|U\|_{H^s(\mathbb{R})} \|\phi'\|_{H^{-s}(\mathbb{R})} \\ &\leq C \|u\|_{H^s(I)} \|\phi\|_{H^{1-s}(\mathbb{R})} \quad \text{(Exercise 1)} \\ &= C \|u\|_{H^s(I)} \|\phi\|_{\widetilde{H}^{1-s}(I)} \ . \end{aligned} \tag{2.4}$$

  Recall the density of test functions in $\widetilde{H}^{1-s}(I)$ and the fact that $H^{s-1}(I)$ is isomorphic with the dual of $\widetilde{H}^{1-s}(I)$.

- Next we show injectivity. Let $u' = 0$. It is sufficient to show that,

  $$< u, \phi >= \int_I u\phi \tag{2.5}$$

vanishes for all test functions with zero average. Indeed, an arbitrary test function can always be decomposed into a constant and a function with zero average,

$$\phi = c + \phi_0, \quad c = \int_I \phi, \quad \int_I \phi_0 = 0\,. \tag{2.6}$$

Then,

$$< u, \phi >= \int_I u(c + \phi_0) = c \int_I u+ < u, \phi_0 >=< u, \phi_0 >\,, \tag{2.7}$$

since we have restricted the derivative operator to functions of zero average. Next,

$$\psi(x) = \int_0^x \phi_0(t)\, dt\,, \tag{2.8}$$

is also a test function and $\psi' = \phi_0$. Thus,

$$< u, \phi_0 >=< u, \psi' >= 0\,, \tag{2.9}$$

since $u' = 0$.

- We show surjectivity by constructing a continuous right inverse. For $s = 1$ we need to integrate simply the derivative. Let $v \in L^2(I)$. Define,

$$u(x) = \int_0^x v(t)\, dt, \quad u_0 = u - \int_I u\,. \tag{2.10}$$

Obviously, $u_0' = v$ and $\|u_0\|_{H^1(I)} \leq C\|v\|_{L^2(I)}$. For $s = 0$ we utilize the following characterization of space $H^{-1}(I)$ [29, 74].

$$H^{-1}(I) = \{v = u_1' + v_1 \;:\; u_1, v_1 \in L^2(I)\}\,, \tag{2.11}$$

with the norm defined by taking the infimum over all possible (non-unique) decompositions of v,

$$\|v\|_{H^{-1}(I)} = \inf_{u_1, v_1} \left(\|u_1\|_{L^2(I)} + \|v_1\|_{L^2(I)}\right)\,. \tag{2.12}$$

We can define then the right-inverse by setting,

$$u = u_1 + \int_0^x v_1, \quad u_0 = u - \int_I u\,. \tag{2.13}$$

Again, u_0 depends continuously upon v in the right norms. By the interpolation argument, the right-inverse can be extended to $H^{s-1}(I)$, for an arbitrary $0 \leq s \leq 1$.

∎

Exercise 1 Show that,

$$\|u'\|_{H^{s-1}(\mathbb{R})} \leq \|u\|_{H^s(\mathbb{R})} \quad \forall s \in \mathbb{R} \tag{2.14}$$

∎

We introduce now the corresponding polynomial exact sequence,

$$\mathbb{R} \to \mathcal{P}^p(I) \xrightarrow{\partial} \mathcal{P}^{p-1}(I) \to \{0\}\ , \tag{2.15}$$

where $\mathcal{P}^p(I)$ denotes the space of polynomials of order less or equal p, defined on unit interval I. In the next section, we shall study various projection operators P_i and projection-based interpolation operators Π_i that make the following diagram commute,

$$\begin{array}{ccccccc}
\mathbb{R} & \to & H^s & \xrightarrow{\partial} & H^{s-1} & \to & \{0\} \\
\downarrow & & P_1 \big\downarrow \Pi_1 & & P_2 \big\downarrow \Pi_2 & & \downarrow \\
\mathbb{R} & \to & \mathcal{P}^p(I) & \xrightarrow{\partial} & \mathcal{P}^{p-1}(I) & \to & \{0\}
\end{array} \tag{2.16}$$

The projections operators P_i will always be defined on the whole spaces but interpolation operators Π_i may be defined only on a subspace due to increased regularity requirements necessary to define e.g. function values at vertices, or average of a function over the integral. We shall also abbreviate the notation by dropping the constants and the trivial spaces, with the understanding however that all properties resulting from the presence of these spaces (surjectivity of ∂, $\mathcal{N}(\partial) = \mathbb{R}$, preservation of constants by P_1, Π_1) are satisfied.

$$\begin{array}{ccc}
H^s & \xrightarrow{\partial} & H^{s-1} \\
\big\downarrow \Pi & & \big\downarrow P \\
\mathcal{P}^p(I) & \xrightarrow{\partial} & \mathcal{P}^{p-1}(I)
\end{array} \tag{2.17}$$

2.2 Two-dimensional sequences

Let Ω be a master triangle,

$$\Omega = \{(x_1, x_2)\ :\ x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\} \tag{2.18}$$

or master square $\Omega = (0,1)^2$. We shall study the following exact sequence,

$$\mathbb{R} \to H^s(\Omega) \xrightarrow{\nabla} \boldsymbol{H}^{s-1}(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} H^{s-1}(\Omega) \to \{0\} \tag{2.19}$$

where ∇ is the gradient operator, curl denotes the scalar-valued curl operator,

$$\mathrm{curl}\, \boldsymbol{E} = E_{1,2} - E_{2,1}\ , \tag{2.20}$$

and $\boldsymbol{H}^{s-1}(\mathrm{curl}, \Omega)$ denotes the subspace of vector fields with both components in $H^{s-1}(\Omega)$ such that the curl is in $H^{s-1}(\Omega)$,

$$\boldsymbol{H}^{s-1}(\mathrm{curl}, \Omega) = \{\boldsymbol{E} \in \boldsymbol{H}^{s-1}(\Omega)\ :\ \mathrm{curl}\, \boldsymbol{E} \in H^{s-1}(\Omega)\}\ . \tag{2.21}$$

We shall restrict ourselves to the range $s \geq \frac{1}{2}$. We will introduce in Section 4.3 a right-inverse of the curl operator demonstrating that the curl operator is a surjection.

Exercise 2 Follow the first step in the proof of Proposition 1 to prove that the gradient operator is well defined. ∎

The two-dimensional exact sequence can be reproduced with several families of polynomials.

Nédélec's triangle of the second type [32]. We have an obvious exact sequence,

$$\mathbb{R} \to \mathcal{P}^p \xrightarrow{\nabla} \boldsymbol{P}^{p-1} \xrightarrow{\text{curl}} \mathcal{P}^{p-2} \to \{0\} \,. \tag{2.22}$$

Here $\mathcal{P}^p$ denotes the space of polynomials of (group) order less or equal p, e.g. $x_1^2 x_2^3 \in \mathcal{P}^5$, and $\boldsymbol{P}^p = \mathcal{P}^p \times \mathcal{P}^p$. Obviously, the construction starts with $p \geq 2$, i.e. the $\boldsymbol{H}(\text{curl})$-conforming elements are at least of first order.

The construction can be generalized to triangles of *variable order*. With each triangle's edge we associate the corresponding edge order p_e. We assume that,

$$p_e \leq p \quad \text{for every edge } e \,.$$

We introduce now the following polynomial spaces:

- The space of scalar-valued polynomials u of order less or equal p, whose traces on edges e reduce to polynomials of (possibly smaller) order p_e,

$$\mathcal{P}^p_{p_e} = \{u \in \mathcal{P}^p \;:\; u|_e \in \mathcal{P}^{p_e}(e)\}.$$

- The space of vector-valued polynomials $\boldsymbol{E}$ of order less or equal p, whose *tangential traces* $E_t|_e$ on edges e reduce to polynomials of order p_e,

$$\boldsymbol{P}^p_{p_e} = \{\boldsymbol{E} \in \boldsymbol{P}^p \;:\; E_t|_e \in \mathcal{P}^{p_e}(e)\}.$$

- The space of scalar-valued polynomials of order less or equal p, with zero average

$$\mathcal{P}^p_{avg} = \{u \in \mathcal{P}^p \;:\; \int_T u \; = 0\}.$$

We have then the exact sequence,

$$\mathcal{P}^p_{p_e} \quad \xrightarrow{\nabla} \quad \boldsymbol{P}^{p-1}_{p_e-1} \quad \xrightarrow{\nabla\times} \quad \mathcal{P}^{p-2}(\mathcal{P}^{p-2}_{avg}) \,. \tag{2.23}$$

The case $p_e = -1$ corresponds to the homogeneous Dirichlet boundary condition. In the case of homogeneous Dirichlet boundary conditions imposed on *all* edges e, the last space in the sequence, corresponding to polynomials of order $p-2$, must be replaced with the space of polynomials with zero average.

Exercise 3 Prove that (2.23) is an exact sequence. ∎

Nédélec's rectangle of the first type [31]. All spaces are defined on the unit square. We introduce the following polynomial spaces.

$$\begin{aligned} W_p &= Q^{(p,q)} \,, \\ \boldsymbol{Q}_p &= Q^{(p-1,q)} \times Q^{(p,q-1)} \,, \\ Y_p &= Q^{(p-1,q-1)} \,. \end{aligned} \tag{2.24}$$

Here, $Q^{p,q} = \mathcal{P}^p \otimes \mathcal{P}^q$ denotes the space of polynomials of order less or equal p, q with respect to x, y, respectively. For instance, $2x^2y^3 \in Q^{(2,3)}$. The polynomial spaces form again an exact sequence,

$$W_p \xrightarrow{\nabla} \boldsymbol{Q}_p \xrightarrow{\nabla\times} Y_p \,. \tag{2.25}$$

The generalization to variable order elements is a little less straightforward than for the triangles. For each horizontal edge e, we introduce order p_e, and with each vertical edge e, we associate order q_e. We assume again that the minimum rule holds, i.e.

$$p_e \le p, \quad q_e \le q \,. \tag{2.26}$$

By $Q^{(p,q)}_{p_e,q_e}$ we understand the space of polynomials of order less or equal p with respect to x and order less or equal q with respect to y, such that their traces to horizontal edges e reduce to polynomials of (possibly smaller than p) degree p_e, and restrictions to vertical edges reduce to polynomials of (possibly smaller than q) order q_e,

$$\begin{aligned} Q^{(p,q)}_{p_e,q_e} = \{u \in Q^{(p,q)} \;:\; & u(\cdot,0) \in \mathcal{P}^{p_1}(0,1), u(\cdot,1) \in \mathcal{P}^{p_2}(0,1), \\ & u(0,\cdot) \in \mathcal{P}^{q_1}(0,1), u(1,\cdot) \in \mathcal{P}^{q_2}(0,1)\} \,. \end{aligned} \tag{2.27}$$

With spaces

$$\begin{aligned} W_p &= Q^{(p,q)}_{p_e,q_e} \,, \\ \boldsymbol{Q}_p &= Q^{(p-1,q)}_{p_e-1} \times Q^{(p,q-1)}_{q_e-1} \,, \\ Y_p &= Q^{(p-1,q-1)} \,, \end{aligned} \tag{2.28}$$

we have the exact sequence,

$$W_p \xrightarrow{\nabla} \boldsymbol{Q}_p \xrightarrow{\nabla\times} Y_p \,. \tag{2.29}$$

Notice that space $\boldsymbol{Q}_p$ cannot be obtained by merely differentiating polynomials from $Q^{(p,q)}_{p_e,q_e}$. For the derivative in x, this would lead to space $Q^{(p-1,q)}_{p_e-1,q_e}$ for the first component, whereas in our definition above q_e has been increased to q. This is motivated by the fact that the traces of E_1 along the *vertical edges* are interpreted as *normal* components of the $\boldsymbol{E}$ field. The $\boldsymbol{H}(\mathrm{curl})$-conforming fields "connect" only through tangential components and, therefore, shape functions corresponding to the normal components on the boundary are classified as interior modes, and they should depend only on the order of the element and not on the order of neighboring elements.

Exercise 4 Prove that (2.29) is an exact sequence. ∎

Nédélec's triangle of the first type [31]. There is a significant difference between the triangular and square elements presented so far. For the triangle, the order p drops upon differentiation from p to $p-2$, see the exact sequence (2.22). This merely reflects the fact that differentiation always lowers the polynomial order by one. In the case of the rectangular element and the Q-spaces, however, the order in the diagram has dropped only by one, from (p,q) to $(p-1,q-1)$, comp. exact sequence (2.25). A similar effect can be obtained for triangles. We shall discuss the concept within the general context of the variable order element.

The goal is to switch from $p-2$ to $p-1$ in the last space in sequence (2.23) *without* increasing the order p in the first space in the sequence. We begin by rewriting (2.23) with p increased by one.

$$\mathcal{P}^{p+1}_{p_e} \quad \xrightarrow{\nabla} \quad \mathbf{P}^{p}_{p_e-1} \quad \xrightarrow{\nabla\times} \quad \mathcal{P}^{p-1} \,. \tag{2.30}$$

Notice that we have not increased the order along the edges. This is motivated with the fact that the edge orders do not affect the very last space in the diagram [1]. Next, we decompose the space of potentials into the previous space of polynomials $\mathcal{P}^{p}_{p_e}$ and *an algebraic complement* $\widetilde{\mathcal{P}}^{p+1}_{p_e}$,

$$\mathcal{P}^{p+1}_{p_e} = \mathcal{P}^{p}_{p_e} \oplus \widetilde{\mathcal{P}}^{p+1}_{p_e} \,. \tag{2.31}$$

The algebraic complement is *not unique*, it may be constructed in (infinitely) many different ways. The decomposition in the space of potentials implies a corresponding decomposition in the $\boldsymbol{H}(\mathrm{curl})$-conforming space,

$$\boldsymbol{P}^{p}_{p_e-1} = \boldsymbol{P}^{p-1}_{p_e-1} \oplus \nabla(\widetilde{\mathcal{P}}^{p+1}_{p_e}) \oplus \widetilde{\boldsymbol{P}}^{p}_{p_e-1} \,. \tag{2.32}$$

The algebraic complement $\widetilde{\boldsymbol{P}}^{p}_{p_e-1}$ is again *not unique*. The desired extension of the original sequence can now be constructed by removing the gradients of order $p+1$,

$$\mathcal{P}^{p}_{p_e} \quad \xrightarrow{\nabla} \quad \mathbf{P}^{p-1}_{p_e-1} \oplus \widetilde{\boldsymbol{P}}^{p}_{p_e-1} \quad \xrightarrow{\nabla\times} \quad \mathcal{P}^{p-1} \,. \tag{2.33}$$

[1]Except for the case of the homogeneous Dirichlet boundary condition imposed on the whole boundary which forces the use of polynomials of zero average for the last space in the diagram

Exercise 5 Prove that (2.33) is an exact sequence. ∎

Note the following facts:

- The modified sequence (2.33) enables the $\boldsymbol{H}(\mathrm{curl})$-conforming discretization of lowest order on triangles. For $p = p_e = 1$,
$$\mathbf{P}^0_0 \oplus \widetilde{\boldsymbol{P}}^1_0 = \boldsymbol{P}^1_0, \quad \dim \boldsymbol{P}^1_0 = 3\,. \tag{2.34}$$
The complement $\widetilde{\mathcal{P}}^2_1$ *is empty* and, therefore, in this case, the resulting space $\boldsymbol{P}^1_0 = \mathbf{P}^0_0 \oplus \widetilde{\boldsymbol{P}}^1_0$, corresponding to the famous construction of Whitney [39], *is unique*. This is the smallest space to enforce the continuity of the (constant) tangential component of $\boldsymbol{E}$ across the interelement boundaries.

- It is not necessary but natural to construct the complements using spans of scalar and vector *bubble functions*. In this case the notation $\widetilde{\mathcal{P}}^{p+1}_{-1}$ and $\widetilde{\boldsymbol{P}}^p_{-1}$ is more appropriate. The concept is especially natural if one uses *hierarchical shape functions*. We can always enforce the zero trace condition by augmenting original shape functions with functions of lower order. In other words, we change the complement but *do not alter* the ultimate polynomial space.

The choice of the complements may be made unique by imposing additional conditions. Nédélec's original construction for elements of uniform order p employs skewsymmetric polynomials,

$$\boldsymbol{R}^p = \{\boldsymbol{E} \in \boldsymbol{P}^p \;:\; \boldsymbol{\epsilon}^p(\boldsymbol{E}) = \mathbf{0}\}\,, \tag{2.35}$$

where $\boldsymbol{\epsilon}^p$ is the Nédélec *symmetrization operator*,

$$\begin{aligned}&(\boldsymbol{\epsilon}^p(\boldsymbol{E}))_{i_1,\ldots,i_{p+1}} \\ &= \frac{1}{p+1}\left(\frac{\partial^p E_{i_1}}{\partial x_{i_2}\ldots\partial x_{i_p}\partial x_{i_{p+1}}} + \frac{\partial^p E_{i_2}}{\partial x_{i_3}\ldots\partial x_{i_{p+1}}\partial x_{i_1}} + \ldots + \frac{\partial^p E_{i_{p+1}}}{\partial x_{i_1}\ldots\partial x_{i_{p-1}}\partial x_{i_p}}\right)\,.\end{aligned} \tag{2.36}$$

The algebraic complement can then be selected as the subspace of *homogeneous*[2] symmetric polynomials $\boldsymbol{D}^p$,

$$\boldsymbol{R}^p = \boldsymbol{P}^p \oplus \boldsymbol{D}^p\,. \tag{2.37}$$

There are many equivalent conditions characterizing the space $\boldsymbol{D}^p$. The most popular one reads as follows

$$\boldsymbol{E} \in \boldsymbol{D}^p \Leftrightarrow \boldsymbol{E} \text{ is homogeneous and } \boldsymbol{x}\cdot\boldsymbol{E}(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x}\,. \tag{2.38}$$

The space $\boldsymbol{D}^p$ can also nicely be characterized as the image of homogeneous polynomials of order $p-1$ under the Poincaré map, see [26, 27],

$$\begin{aligned} E_1(\boldsymbol{x}) &= -x_2 \int_0^1 t\psi(t\boldsymbol{x})\,dt \\ E_2(\boldsymbol{x}) &= x_1 \int_0^1 t\psi(t\boldsymbol{x})\,dt\,. \end{aligned} \tag{2.39}$$

[2] A polynomial of order p is homogeneous if it can be represented as a sum of monomials of order p. Equivalently, $u(\xi x_1,\ldots,\xi x_n) = \xi^p u(x_1,\ldots,x_n)$.

The Poincare map is a right inverse of the curl map, $\nabla \times \boldsymbol{E} = \psi$, for the $\boldsymbol{E}$ defined above. Consistently with our discussion, it can be shown that the tangential trace of a symmetric polynomial of order p is always a polynomial of order less or equal $p-1$. For other characterizations of the space $\boldsymbol{D}^p$, see [23]. An important property of the Nédélec space $\boldsymbol{R}^p$ is that it is invariant under affine transformations, comp. Exercise 6. Consequently, the polynomial space is independent of the way in which the vertices of the triangle are enumerated.

Exercise 6 Prove that Nédélec's space is affine invariant. More precisely, let $\boldsymbol{x} \to \boldsymbol{y} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{b}$ denote a non-singular affine map from $\mathbb{R}^n$ into itself. Let $\hat{\boldsymbol{E}} = \hat{\boldsymbol{E}}(\boldsymbol{x})$ be a symmetric polynomial of order p, i.e.,

$$\boldsymbol{x} \cdot \hat{\boldsymbol{E}}(\boldsymbol{x}) = 0, \quad \forall \boldsymbol{x} \, . \tag{2.40}$$

Define,

$$E_i(\boldsymbol{y}) = \sum_j \hat{E}_j(\boldsymbol{x}) \frac{\partial x_j}{\partial y_i}, \quad \boldsymbol{y} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{b} \, . \tag{2.41}$$

Show that

$$\boldsymbol{y} \cdot \boldsymbol{E}(\boldsymbol{y}) = 0, \quad \forall \boldsymbol{y} \, . \tag{2.42}$$

∎

Uniqueness of the spaces could also be naturally enforced by requesting *orthogonality* of algebraic complements [38, 15],

$$\begin{aligned} \mathcal{P}^{p+1}_{p_e} = \mathcal{P}^{p}_{p_e} \oplus \widetilde{\mathcal{P}}^{p+1}_{-1}, \quad \mathcal{P}^{p+1}_{-1} = \mathcal{P}^{p}_{-1} \overset{\perp}{\oplus} \widetilde{\mathcal{P}}^{p+1}_{-1} \\ \boldsymbol{P}^{p}_{p_e-1} = \boldsymbol{P}^{p-1}_{p_e-1} \oplus \nabla(\widetilde{\mathcal{P}}^{p+1}_{p_e}) \oplus \widetilde{\boldsymbol{P}}^{p}_{-1}, \quad \boldsymbol{P}^{p}_{-1} = \boldsymbol{P}^{p-1}_{-1} \overset{\perp}{\oplus} \nabla(\widetilde{\mathcal{P}}^{p+1}_{-1}) \overset{\perp}{\oplus} \widetilde{\boldsymbol{P}}^{p}_{-1} \, . \end{aligned} \tag{2.43}$$

The orthogonality for the scalar-valued and the vector-valued polynomial spaces is usually understood in the sense of H^1_0 and $\boldsymbol{H}(\mathrm{curl})$ scalar products, respectively.

Parametric elements. The concept of an exact sequence of discrete (finite-dimensional) spaces goes beyond polynomial spaces. Study of the construction of the parametric element and the corresponding exact sequence is not only necessary for dealing with curved elements but it enhances essentially the understanding of the polynomial spaces, e.g. the concept of affine and "rotational" invariance. We will discuss the notion of parametric elements after we present the 3D exact polynomial sequences.

Nédélec tetrahedron of the second type [32]. All polynomial spaces are defined on the master tetrahedron,

$$\Omega = \{(x_1, x_2, x_3) \; : \; x_1 > 0, x_2 > 0, x_3 > 0, x_1 + x_2 + x_3 < 1\} \, . \tag{2.44}$$

We have the following exact sequence,

$$\mathcal{P}^p \xrightarrow{\nabla} \boldsymbol{P}^{p-1} \xrightarrow{\nabla\times} \boldsymbol{P}^{p-2} \xrightarrow{\nabla\circ} \mathcal{P}^{p-3} . \tag{2.45}$$

Here $\mathcal{P}^p$ denotes the space of polynomials of (group) order less or equal p, e.g. $x_1^2x_2^3x_3^2 \in \mathcal{P}^7$, and $\boldsymbol{P}^p = \mathcal{P}^p \times \mathcal{P}^p \times \mathcal{P}^p$. Obviously, the construction starts with $p \geq 3$, i.e. the $\boldsymbol{H}(\mathrm{curl})$-conforming elements are at least of second order.

The construction can be generalized to tetrahedra of *variable order*. With each tetrahedron's face we associate the corresponding face order p_f, and with each tetrahedron's edge, we associate the corresponding edge order p_e. We assume that,

$$p_f \leq p \,\forall \text{ face } f, \quad p_e \leq p_f \forall \text{ face } f \text{ adjacent to edge } e \text{ , } \forall \text{ edge } e \text{ .} \tag{2.46}$$

The assumption is satisfied in practice by enforcing the *minimum rule*, i.e. setting the face and edge orders to the minimum of the orders of the adjacent elements. We introduce now the following polynomial spaces.

- The space of scalar-valued polynomials of order less or equal p, whose traces on faces f reduce to polynomials of (possibly smaller) order p_f, and whose traces on edges e reduce to polynomials of (possibly smaller) order p_e,

$$\mathcal{P}^p_{p_f,p_e} = \{u \in \mathcal{P}^p \;:\; u|_f \in \mathcal{P}^{p_f}(f),\; u|_e \in \mathcal{P}^{p_e}(e)\}. \tag{2.47}$$

- The space of vector-valued polynomials of order less or equal p, whose *tangential traces* on faces f reduce to polynomials of order p_f, and whose *tangential traces* on edges e reduce to polynomials of order p_e,

$$\boldsymbol{P}^p_{p_f,p_e} = \{\boldsymbol{E} \in \boldsymbol{P}^p \;:\; \boldsymbol{E}_t|_f \in \boldsymbol{P}^{p_f}(f),\; E_t|_e \in \mathcal{P}^{p_e}(e)\}. \tag{2.48}$$

- The space of vector-valued polynomials of order less or equal p, whose *normal traces* on faces f reduce to polynomials of order p_f

$$\boldsymbol{P}^p_{p_f} = \{\boldsymbol{E} \in \boldsymbol{P}^p \;:\; E_n|_f \in \mathcal{P}^{p_f}(f)\}. \tag{2.49}$$

We have then the exact sequence,

$$\mathcal{P}^p_{p_f,p_e} \xrightarrow{\nabla} \boldsymbol{P}^{p-1}_{p_f-1,p_e-1} \xrightarrow{\nabla\times} \boldsymbol{P}^{p-2}_{p_f-2} \xrightarrow{\nabla\circ} \mathcal{P}^{p-3} , \tag{2.50}$$

The case $p_f, p_e = -1$ corresponds to the homogeneous Dirichlet boundary condition.

Nédélec's hexahedron of the first type [31]. All polynomial spaces are defined on a unit cube. We introduce the following polynomial spaces.

$$\begin{aligned} W_p &= Q^{(p,q,r)} \\ \boldsymbol{Q}_p &= Q^{(p-1,q,r)} \times Q^{(p,q-1,r)} \times Q^{(p,q,r-1)} \\ \boldsymbol{V}_p &= Q^{(p,q-1,r-1)} \times Q^{(p-1,q,r-1)} \times Q^{(p-1,q-1,r)} \\ Y_p &= Q^{(p-1,q-1,r-1)} \,. \end{aligned} \tag{2.51}$$

Here $Q^{p,q,r} = \mathcal{P}^p \otimes \mathcal{P}^q \otimes \mathcal{P}^r$ denotes the space of polynomials of order less or equal p, q, r with respect to x, y, z, respectively. For instance, $2x^2y^3 + 3x^3z^8 \in Q^{(3,3,8)}$. The polynomial spaces form again the exact sequence,

$$W_p \xrightarrow{\nabla} \boldsymbol{Q}_p \xrightarrow{\nabla\times} \boldsymbol{V}_p \xrightarrow{\nabla\circ} Y_p \,. \tag{2.52}$$

The generalization to variable order elements is a little less straightforward than for the tetrahedra. Review the 2D construction first. In three dimensions, spaces get more complicated and notation more cumbersome. We start with the space,

$$Q^{(p,q,r)}_{(p_f,q_f),(p_f,r_f),(q_f,r_f),p_e,q_e,r_e} \,, \tag{2.53}$$

that consists of polynomials in $Q^{(p,q,r)}$ such that:

- their restrictions to faces f parallel to axes x, y reduce to polynomials in $Q^{(p_f,q_f)}$,
- their restrictions to faces f parallel to axes x, z reduce to polynomials in $Q^{(p_f,r_f)}$,
- their restrictions to faces f parallel to axes y, z reduce to polynomials in $Q^{(q_f,r_f)}$,
- their restriction to edges parallel to axis x, y, z reduce to polynomials of order p_e, q_e, r_e respectively,

with the minimum rule restrictions:

$$p_f \le p, q_f \le q, r_f \le r, \quad p_e \le p_f, q_e \le q_f, r_e \le r_f, \text{ for adjacent faces } f \,. \tag{2.54}$$

The 3D polynomial spaces forming the de Rham diagram, are now introduced as follows,

$$\begin{aligned} W_p &= Q^{(p,q,r)}_{(p_f,q_f),(p_f,r_f),(q_f,r_f),p_e,q_e,r_e} \\ \boldsymbol{Q}_p &= Q^{(p-1,q,r)}_{(p_f-1,q_f),(p_f-1,r_f),p_e-1,q_f,r_f} \times Q^{(p,q-1,r)}_{(p_f,q_f-1),(q_f-1,r_f),p_f,q_e-1,r_f} \\ &\quad \times Q^{(p,q,r-1)}_{(p_f,r_f-1),(q_f,r_f-1),p_f,q_f,r_e-1} \\ \boldsymbol{V}^p &= Q^{(p,q-1,r-1)}_{(q_f-1,r_f-1)} \times Q^{(p-1,q,r-1)}_{(p_f-1,r_f-1)} \times Q^{(p-1,q-1,r)}_{(p_f-1,q_f-1)} \\ Y_p &= Q^{(p-1,q-1,r-1)} \,. \end{aligned} \tag{2.55}$$

Note the following points:

- There is no restriction on edge order in the $\boldsymbol{H}(\mathrm{div})$ -conforming space. The only order restriction is placed on faces normal to the particular component, e.g. for the first component H_1, the order restriction is imposed only on faces parallel to y, z faces.

- For the $\boldsymbol{H}(\mathrm{curl})$-conforming space, there is no restriction on face order for faces perpendicular to the particular component. For instance, for E_1, there is no order restriction on faces parallel to y, z axes. The edge orders for edges perpendicular to x are inherited from faces *parallel* to the x axis. This is related to the fact that elements connecting through the first component E_1, connect only through faces and edges parallel to the first axis only.

Exercise 7 Prove that the spaces defined above form an exact sequence. ∎

Nédélec tetrahedron of the first type [31]. Review the construction of the corresponding triangular element first. The 3D construction goes along the same lines but it becomes more technical. We discuss the element of variable order. The following decompositions are relevant.

$$\begin{aligned}
\mathcal{P}^{p+1}_{p_e,p_f+1} &= \mathcal{P}^{p}_{p_e,p_f} \oplus \widetilde{\mathcal{P}}^{p+1}_{-1,p_f+1} \\
\boldsymbol{P}^{p}_{p_e-1,p_f} &= \boldsymbol{P}^{p-1}_{p_e-1,p_f-1} \oplus \nabla(\widetilde{\mathcal{P}}^{p+1}_{-1,p_f+1}) \oplus \widetilde{\boldsymbol{P}}^{p}_{-1,p_f} \\
\boldsymbol{P}^{p}_{p_f} &= \boldsymbol{P}^{p-1}_{p_f-1} \oplus \nabla(\widetilde{\boldsymbol{P}}^{p+1}_{-1,p_f+1}) \oplus \widetilde{\boldsymbol{P}}^{p}_{-1}
\end{aligned} \tag{2.56}$$

The ultimate sequence looks as follows:

$$\mathcal{P}^{p}_{p_e,p_f} \xrightarrow{\nabla} \boldsymbol{P}^{p-1}_{p_e-1,p_f-1} \oplus \widetilde{\boldsymbol{P}}^{p}_{-1,p_f} \xrightarrow{\nabla\times} \boldsymbol{P}^{p-1}_{p_f-1} \oplus \widetilde{\boldsymbol{P}}^{p}_{-1} \xrightarrow{\nabla\circ} \mathcal{P}^{p-1} \,. \tag{2.57}$$

Referring to [38, 15] for details, we emphasize only that switching to the tetrahedra of the first type in 3D, requires adding not only extra interior bubbles but face bubbles as well. The actual construction of Nédélec involves the choice of a special complement $\widetilde{\boldsymbol{P}}^{p}_{-1,p_f}$ consisting of antisymmetric polynomials; all remarks on the 2D element, including a characterization using Poincare's maps, remain valid.

Exercise 8 Prove that the spaces defined above form an exact sequence. ∎

Prismatic elements. We shall not discuss here the construction of the exact sequences for the prismatic elements. The prismatic element shape functions are constructed as tensor products of triangular element and 1D element shape functions. We can use both Nedelec's triangles for the construction and, consequently, we can also produce two corresponding exact sequences.

Parametric elements. Given a bijective map $\boldsymbol{x} = \boldsymbol{x}_{\Omega}(\boldsymbol{\xi})$ transforming master element $\hat{\Omega}$ onto a physical element Ω, and master element shape functions $\hat{\phi}(\boldsymbol{\xi})$, we define the H^1-conforming shape functions on the physical element in terms of master element coordinates,

$$\phi(\boldsymbol{x}) = \hat{\phi}(\boldsymbol{\xi}) = \hat{\phi}(\boldsymbol{x}_\Omega^{-1}(\boldsymbol{x})) = (\hat{\phi} \circ \boldsymbol{x}_\Omega^{-1})(\boldsymbol{x}) \, . \tag{2.58}$$

The definition reflects the fact that the integration of master element matrices is always done in terms of master element coordinates and, therefore, it is simply convenient to define the shape functions in terms of master coordinates $\boldsymbol{\xi}$. This implies that the parametric element shape functions are compositions of the inverse $\boldsymbol{x}_\Omega^{-1}$ and the master element polynomial shape functions. In general, we do not deal with polynomials anymore. In order to keep the exact sequence property, we have to define the $\boldsymbol{H}(\mathrm{curl})$-, $\boldsymbol{H}(\mathrm{div})$-, and L^2-conforming elements consistently with the way the differential operators transform. For gradients we have,

$$\frac{\partial u}{\partial x_i} = \frac{\partial \hat{u}}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i} \tag{2.59}$$

and, therefore,

$$E_i = \hat{E}_k \frac{\partial \xi_k}{\partial x_i} \, . \tag{2.60}$$

For the curl operator we have,

$$\epsilon_{ijk}\frac{\partial E_k}{\partial x_j} = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\hat{E}_l \frac{\partial \xi_l}{\partial x_k}\right) = \epsilon_{ijk}\frac{\partial \hat{E}_l}{\partial x_j}\frac{\partial \xi_l}{\partial x_k} + \hat{E}_l \underbrace{\epsilon_{ijk}\frac{\partial^2 \xi_l}{\partial x_k \partial x_j}}_{=0} = \epsilon_{ijk}\frac{\partial \hat{E}_l}{\partial \xi_m}\frac{\partial \xi_m}{\partial x_j}\frac{\partial \xi_l}{\partial x_k} \, . \tag{2.61}$$

But,

$$\epsilon_{ijk}\frac{\partial \xi_m}{\partial x_j}\frac{\partial \xi_l}{\partial x_k} = J^{-1}\epsilon_{nml}\frac{\partial x_i}{\partial \xi_n} \, , \tag{2.62}$$

where J^{-1} is the inverse jacobian. Consequently,

$$\epsilon_{ijk}\frac{\partial E_k}{\partial x_j} = J^{-1}\frac{\partial x_i}{\partial \xi_n}\left(\epsilon_{nml}\frac{\partial \hat{E}_l}{\partial \xi_m}\right) \, . \tag{2.63}$$

This leads to the definition of the $\boldsymbol{H}(\mathrm{div})$-conforming parametric element,

$$H_i = J^{-1}\frac{\partial x_i}{\partial \xi_n}\hat{H}_n \, . \tag{2.64}$$

Finally,

$$\frac{\partial H_i}{\partial x_i} = \underbrace{\frac{\partial}{\partial x_i}\left(J^{-1}\frac{\partial x_i}{\partial \xi_k}\right)}_{=0}\hat{H}_k + J^{-1}\frac{\partial x_i}{\partial \xi_k}\frac{\hat{H}_k}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_i} = J^{-1}\frac{\partial \hat{H}_k}{\partial \xi_k} \, , \tag{2.65}$$

which establishes the transformation rule for the L^2-conforming elements,

$$f = J^{-1}\hat{f} \, . \tag{2.66}$$

Defining the parametric element spaces $W_p, \boldsymbol{Q}_p, \boldsymbol{V}_p, Y_p$ using the transformation rules listed above, we preserve for the parametric element the exact sequence (2.52).

In the case of the *isoparametric element*, the components of the transformation map $\boldsymbol{x}_\Omega$ come from the space of the H^1-conforming master element,

$$x_j = \sum_k x_{j,k}\hat{\phi}_k(\boldsymbol{\xi}) = \sum_k \boldsymbol{x}_k \phi_k(\boldsymbol{x}) \ .$$

Here $x_{j,k}$ denote the (vector-valued) geometry degrees-of-freedom corresponding to element shape functions $\phi_k(\boldsymbol{x})$. By construction, therefore, the parametric element shape functions can reproduce any linear function $a_j x_j$. As they also can reproduce constants, the isoparametric element space of shape functions contains the space of all linear polynomials in $\boldsymbol{x}$ - $a_j x_j + b$, in mechanical terms - the space of linearized rigid body motions. The exact sequence property implies that the $\boldsymbol{H}(\mathbf{curl})$-conforming element can reproduce only constant fields, but the $\boldsymbol{H}(\mathrm{div})$-conforming element, in general, cannot reproduce even constants. This indicates in particular that, in context of general parametric (non affine) elements [3] unstructured mesh generators should be used with caution, comp. [2]. The critique does not apply to (algebraic) mesh generators based on a consistent representation of the domain as a manifold, with underlying global maps parametrizing portions of the domain. Upon a change of variables, the original problem can then be redefined in the reference domain discretized with affine elements, see [17] for more details.

3 Commuting Projections and Projection-Based Interpolation Operators in One Space Dimension

3.1 Commuting projections. Projection error estimates

Let $I = (0, 1)$. We consider the following diagram.

$$\begin{array}{ccccccc}
\mathbb{R} & \longrightarrow & H^s(I) & \xrightarrow{\partial} & H^{s-1}(I) & \longrightarrow & \{0\} \\
\big\downarrow & & \big\downarrow P_s^\partial & & \big\downarrow P_{s-1} & & \big\downarrow \\
\mathbb{R} & \longrightarrow & \mathcal{P}^p & \xrightarrow{\partial} & \mathcal{P}^{p-1} & \longrightarrow & \{0\}
\end{array} \tag{3.67}$$

Here P_{s-1} is the standard orthogonal projection in $H^{s-1}(I)$-norm, and the operator P_s^∂ is defined as follows.

$$\left\{\begin{array}{l}
P_s^\partial u =: u_p \in \mathcal{P}^p(I) \\
\|u_p' - u'\|_{H^{s-1}(I)} \to \min \\
(u_p - u, 1)_{H^s(I)} = 0
\end{array}\right. \tag{3.68}$$

We are interested in the range $0 \le s \le 1$.

Exercise 9 Show that the diagram above commutes. ▮

[3] Note that general quadrilaterals or hexahedra with straight edges are not affi ne elements

Finding the projection $P_s^\partial u$ can be interpreted as the solution of a constrained minimization problem leading to the following mixed formulation.

$$\left\{\begin{array}{llll} P_s^\partial u =: u_p \in \mathcal{P}^p(I), & \lambda \in \mathbb{R} & & \\ (u_p' - u', v')_{H^{s-1}(I)} & +(\lambda, v)_{H^s(I)} & = 0 & \forall v \in \mathcal{P}^p(I) \\ (u_p - u, \mu)_{H^s(I)} & & = 0 & \forall \mu \in \mathbb{R} \end{array}\right. \tag{3.69}$$

Here λ is a constant Lagrange multiplier. Substituting $v = \lambda$ in the first equation, we learn that the Lagrange multiplier must be zero. By the Brezzi's theory [10], estimation of the projection error involves the satisfaction of two inf-sup conditions:

- the inf-sup condition relating the space of solutions $\mathcal{P}^p$ and the multiplier space $\mathbb{R}$,

$$\sup_{v \in \mathcal{P}^p} \frac{|(\lambda, v)_{H^s(I)}|}{\|v\|_{H^s(I)}} \geq \beta |\lambda|, \quad \forall \lambda \in \mathbb{R}\,, \tag{3.70}$$

- the inf-sup in kernel condition,

$$\sup_{v \in \mathcal{P}^p_{avg}} \frac{|(u', v')_{H^{s-1}(I)}|}{\|v\|_{H^s(I)}} \geq \alpha \|u\|_{H^s(I)}, \quad \forall u \in \mathcal{P}^p_{avg}\,. \tag{3.71}$$

Notice that (Exercise 10),

$$(u, 1)_{H^s(I)} = (u, 1)_{L^2(I)} = \int_I u\,. \tag{3.72}$$

The first inf-sup condition is a direct consequence of the discrete exact sequence property, i.e. the fact that constants are reproduced by the polynomials, and that (comp.3.72),

$$\|1\|_{H^s(I)} = 1\,. \tag{3.73}$$

The choice of $v = \lambda$ then gives $\beta = 1$. The second inf-sup condition is implied by a Poincare-like inequality,

$$\|u\|_{H^s(I)} \leq C \|u'\|_{H^{s-1}(I)}, \quad \forall u \in H^s(I)\,:\, \int_I u = 0\,. \tag{3.74}$$

This follows immediately from Proposition 1. Notice that we actually need only a discrete version of the inequality but with a constant independent of p.

Exercise 10 Prove (3.72). ∎

We can formulate now our projections errors estimate.

THEOREM 1

There exist constants C, independent[4] of p such that,

$$
\begin{aligned}
\|u - P_s^\partial u\|_{H^s(I)} &\leq C \inf_{w\in\mathcal{P}^p} \|u-w\|_{H^s(I)} &&\leq C p^{-(r-s)} \|u\|_{H^r(I)}, && \forall u \in H^r(I),\\
\|E - P_{s-1}E\|_{H^s(I)} &= \inf_{F\in\mathcal{P}^{p-1}} \|E-F\|_{H^{s-1}(I)} &&\leq C(p-1)^{-(r-s)} \|E\|_{H^{r-1}(I)}, && \forall E \in H^{r-1}(I),
\end{aligned}
\tag{3.75}
$$

for $s < r$. ▮

Proof: The proof of the first estimate follows immediately from Brezzi's theory, standard best approximation error estimates for polynomials [37, p.75],

$$
\begin{aligned}
\inf_{w\in\mathcal{P}^p} \|u-w\|_{L^2(I)} &\leq C p^{-r} \|u\|_{H^r(I)}, & r \geq 1\\
\inf_{w\in\mathcal{P}^p} \|u-w\|_{H^1(I)} &\leq C p^{-(r-1)} \|u\|_{H^r(I)}, & r \geq 1
\end{aligned}
\tag{3.76}
$$

and an interpolation argument. We first interpolate with $0 \leq s \leq 1$ to obtain,

$$
\inf_{w\in\mathcal{P}^p} \|u-w\|_{H^s(I)} \leq C p^{-(r-s)} \|u\|_{H^r(I)}, \quad r \geq 1 \tag{3.77}
$$

and next with r in between s and the original r, to get the final estimate. The second estimate follows from the first one and Proposition 1. ▮

An alternative characterization of P_s^∂. Let K be the inverse of the derivative operator studied in Proposition 1. Operator K is continuous and polynomial preserving. Let P_{s-1} be the orthogonal projection in $H^{s-1}(I)$-norm onto polynomials $\mathcal{P}^{p-1}$, and let P_s^0 be the orthogonal projection in $H^s(I)$-norm[5] onto the null space of the derivative operator, i.e. the constants. Then,

$$
P_s^\partial = K P_{s-1}\partial + P_s^0(I - KP\partial) \tag{3.78}
$$

Consequently,

$$
I - P_s^\partial = (I - P_s^0)(I - KP\partial)\,, \tag{3.79}
$$

and the error estimate follows simply from the continuity of the operator K. The characterization illuminates the role of the inverse operator K.

[4] and s as well

[5] Projection onto constants into H^s norm is equivalent to the L^2-projection

3.2 Commuting interpolation operators. Interpolation error estimates

3.2.1 The range $\frac{1}{2} < s \leq 1$

We consider the following diagram.

$$\begin{array}{ccccccc} \mathbb{R} & \longrightarrow & H^s & \xrightarrow{\partial} & H^{s-1} & \longrightarrow & \{0\} \\ \Big\downarrow id & & \Big\downarrow \Pi_s^{\partial} & & \Big\downarrow \Pi_{s-1} & & \\ \mathbb{R} & \longrightarrow & \mathcal{P}^p & \xrightarrow{\nabla} & \mathcal{P}^{p-1} & \longrightarrow & \{0\} \end{array} \tag{3.80}$$

Here Π_s^{∂} and Π_{s-1} are the projection-based interpolation operators defined as follows.

$$\begin{cases} \Pi_s^{\partial} u =: u_p \in \mathcal{P}^p(I) \\ u_p = u \text{ at } 0, 1 \\ \|u_p' - u'\|_{H^{s-1}(I)} \to \min \end{cases} \tag{3.81}$$

and,

$$\begin{cases} \Pi_{s-1} E =: E_{p-1} \in \mathcal{P}^{p-1}(I) \\ < E_{p-1} - E, 1 >= 0 \\ \|E_{p-1} - E\|_{H^{s-1}(I)} \to \min \end{cases} \tag{3.82}$$

We are restricting ourselves first to $\frac{1}{2} < s \leq 1$. The problem of finding the $\Pi^{\partial} u$-interpolant can again be interpreted as a constrained minimization problems that leads to the following variational characterization,

$$\begin{cases} \Pi_s^{\partial} u =: u_p \in \mathcal{P}^p(I) \\ u_p(0) = u(0), \quad u_p(1) = u(1), \\ (u_p' - u', v')_{H^{s-1}(I)}, \quad \forall v \in \mathcal{P}^p \ : \ v(0) = v(1) = 0 \end{cases} \tag{3.83}$$

Thus, finding the value of the commuting projection operator reduces to the solution of a local Neumann problem, and finding the interpolant is equivalent to a local Dirichlet problem. Similarly, determining $\Pi_{s-1} E$ is equivalent to the variational problem,

$$\begin{cases} \Pi_{s-1} E =: E_{p-1} \in \mathcal{P}^{p-1}(I) \\ < E_{p-1} - E, 1 >= 0, \\ (E_{p-1} - E, v)_{H^{s-1}(I)}, \quad \forall v \in \mathcal{P}^{p-1} \ : \ \int_I v = 0 \end{cases} \tag{3.84}$$

Proposition 2
Diagram above commutes. ∎

Proof: First notice that the interpolation operator Π_s^∂ is well defined and that it preserves constants, i.e. the first part of the diagram commutes. The operator Π_{s-1} is defined on distributions from H^{s-1} with range $-\frac{1}{2} < s - 1 \leq 0$. Constant function 1 belongs to the dual $\widetilde{H}_{1-s},\ 0 \leq 1 - s < \frac{1}{2}$, so the average value is well-defined, comp. Exercise 11. We need to show that for $u \in H^s(I), u(0) = u(1) = 0$,

$$< u', 1 >= 0 \,. \tag{3.85}$$

Let $\phi_n \in \mathcal{D}(I)$ be a sequence of test functions converging to 1 in $\widetilde{H}_{1-s}$-norm. By Proposition 1 and the duality argument, the derivative operator ∂ is a continuous map from $\widetilde{H}^s \to \widetilde{H}^{s-1}$. Consequently $\phi_n' \to 0$ in $\widetilde{H}^{s-1}$-norm. Next, let $\psi_m \in \mathcal{D}(I)$ be a sequence of test functions converging to u in the H^s-norm. Integration by parts yields,

$$< \psi_m', \phi_n >= \int_I \psi_m' \phi_n = - \int_I \psi_m \phi_n' = - < \psi_m, \phi_n' > \tag{3.86}$$

Passing to the limit with n and m, we get the required result. Finally, the orthogonality condition for the derivative implies that,

$$(u_p' - u', v)_{H^{s-1}(I)} = 0, \quad \forall v \in \mathcal{P}^{p-1} \ : \ \int_I v = 0 \tag{3.87}$$

This is a consequence of the fact that the range of the derivative operator restricted to polynomials of order p that vanish at the endpoints, coincides with polynomials of order $p-1$ with zero average. ∎

Exercise 11 Let $u \in H^{-r}(I),\ 0 \leq r < \frac{1}{2}$. Let $\phi_n \in \mathcal{D}(I)$ be a sequence of test functions converging to 1 in $\widetilde{H}_r$-norm, comp. [29, p.77]. Prove that the limit,

$$\lim_{n\to\infty} < u, \phi_n > \tag{3.88}$$

exists, and it is independent of the choice of the sequence. ∎

THEOREM 2
There exist constants C, independent of p such that,

$$\begin{aligned}
\|u - \Pi_s^\partial u\|_{H^s(I)} &\leq C \inf_{w\in\mathcal{P}^p} \|u - w\|_{H^s(I)} &&\leq C p^{-(r-s)} \|u\|_{H^r(I)}, &&\forall u \in H^r(I), \\
\|E - \Pi_{s-1} E\|_{H^{s-1}(I)} &\leq C \inf_{F\in\mathcal{P}^{p-1}} \|E - F\|_{H^{s-1}(I)} &&\leq C(p-1)^{-(r-s)} \|E\|_{H^{r-1}(I)}, &&\forall E \in H^{r-1}(I),
\end{aligned} \tag{3.89}$$

for $\frac{1}{2} < s < r$. With $s = \frac{1}{2} + \epsilon$, and $\epsilon \to 0$, constants $C = O(\epsilon^{-\frac{1}{2}})$. ∎

Lemma 1
Linear extension,

$$u(x) = u_0(1-x) + u_1 x \tag{3.90}$$

defines a continuous extension operator $Ext : \mathbb{R}^2 \ni (u_0, u_1) \to u \in H^s(I)$*, with a norm independent of* s.
∎

Proof: The result follows from obvious cases for $s = 0$ and $s = 1$, and the interpolation argument. ∎

Proof: The main strategy to derive the interpolation error estimates now is to compare the interpolation errors with the projection errors. We begin with the triangle inequality,

$$\|u - \Pi_s^\partial u\|_{H^s(I)} \le \|u - P_s^\partial u\|_{H^s(I)} + \|P_s^\partial u - \Pi_s^\partial u\|_{H^s(I)} \tag{3.91}$$

Polynomial $\psi = P_s^\partial u - \Pi_s^\partial u$ satisfies,

$$(\psi', \phi')_{H^{s-1}(I)} = 0, \quad \forall \phi \in \mathcal{P}^p \ : \ \phi(0) = \phi(1) = 0 \tag{3.92}$$

and, therefore, it is the discrete minimum energy extension with the energy defined by the $H^{s-1}(I)$-norm of derivative ψ'. Moreover, for $s > \frac{1}{2}$, the derivative operator ∂ is an isomorphism mapping $H_0^s(I)$ onto the subspace of $H_0^{s-1}(I)$ consisting of distributions with zero average. Consequently, its inverse is continuous and we have,

$$\|u\|_{H^s(I)} \le C\|u'\|_{H^{s-1}(I)}, \quad \forall u \in H_0^s(I) \tag{3.93}$$

Exercise 12 Prove that the constant C in (3.93) is $C = O(\epsilon^{-\frac{1}{2}})$ for $s = \frac{1}{2} + \epsilon$. ∎

Let ψ_0 be now the minimum-norm polynomial extension of the boundary values of ψ. Then,

$$\begin{aligned}
\|\psi\|_{H^s(I)} &\le \|\psi - \psi_0\|_{H^s(I)} + \|\psi_0\|_{H^s(I)} \\
&\le C(\|\psi' - \psi_0'\|_{H^{s-1}(I)} + \|\psi_0\|_{H^s(I)}) \\
&\le C(\|\psi\prime\|_{H^{s-1}(I)} + \|\psi_0\|_{H^s(I)}) \\
&\le C(\|\psi_0\prime\|_{H^{s-1}(I)} + \|\psi_0\|_{H^s(I)}) \\
&\le C(\|\psi_0\|_{H^s(I)}
\end{aligned} \tag{3.94}$$

Thus, the inequality (3.93) implies that the norm of the polynomial minimum-seminorm extension is always bounded by the norm of the minimum-norm polynomial extension. Denoting the trace of ψ at the end-points of interval I by $tr\psi$, we have now by Lemma 1 and Proposition 1,

$$\begin{aligned} \|\psi\|_{H^s(I)} &\leq C\|Ext\, tr\psi\|_{H^s(I)} \\ &\leq C\|Ext\|\, |\Pi^\partial_s u - P^\partial_s u|_{\partial I} \\ &= C\|Ext\|\, |u - P^\partial_s u|_{\partial I} \\ &\leq CC_{tr}\|Ext\|\, \|u - P^\partial_s u\|_{H^s(I)} \end{aligned} \tag{3.95}$$

Here C_{tr} is the norm of the trace operator of order $O(\epsilon^{-\frac{1}{2}})$ for $s = \frac{1}{2} + \epsilon$, comp. [29, p.100]. Combining the triangle inequality (3.91) with the result above, we see that the interpolation error is bounded by the projection error, and the result follows from Theorem 1.

The estimate for operator Π_{s-1} follows now from the estimate for Π^∂_s and the commutativity argument. Let $E \in H^{r-1}(I)$ and let $u \in H^r(I)$ be the value of the inverse of the derivative operator defined in Proposition 1. Then,

$$\|E - \Pi_{s-1}E\|_{H^{s-1}(I)} = \|(u - \Pi^\partial_s u)'\|_{H^{s-1}(I)} \leq \|u - \Pi^\partial_s u\|_{H^s(I)} \tag{3.96}$$

▮

Remark 1 An alternative proof of the projection estimates for the whole range $0 \leq s \leq 1$, and the interpolation estimates for $\frac{1}{2} < s \leq 1$, follows from the standard argument for continuous, polynomial preserving operators. We have, e.g.

$$\begin{aligned} \|u - \Pi^\partial_s u\|_{H^s(I)} &= \|(u-\phi) - \Pi^\partial_s(u-\phi)\|_{H^s(I)} \qquad (\forall \phi \in \mathcal{P}^p(I)) \\ &\leq \|I - \Pi^\partial_s\|_{\mathcal{L}(H^s,H^s)} \inf_{\phi\in\mathcal{P}^p(I)} \|u-\phi\|_{H^s(I)} \end{aligned} \tag{3.97}$$

Our presentation reflects the strategy that we will use for the two- and the three-dimensional case. ▮

3.2.2 The case $s = \frac{1}{2}$

Contrary to the commuting projection operators that exhibit the best approximation property for the whole range of $s \in [0,1]$, the minimum regularity assumption for the functions being interpolated is $r > \frac{1}{2}$. This does not prohibit defining the projection-based interpolation operators for $s = \frac{1}{2}$. The corresponding interpolation errors, measured in $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$ norms are no longer bounded by the best approximation errors in the same norms. We do get, however, almost optimal p-estimates "polluted" with logarithmic terms only. Repeating argument from the proof of Theorem 2,

$$\begin{aligned} \|\psi'\|_{H^{-\frac{1}{2}}(I)} &\leq \|(Ext\, tr\psi)'\|_{H^{-\frac{1}{2}}(I)} \\ &\leq C\|Ext\|\, |\Pi^\partial_{\frac{1}{2}} u - P^\partial_{\frac{1}{2}} u|_{\partial I} \\ &= C\|Ext\|\, |u - P^\partial_{\frac{1}{2}} u|_{\partial I} \\ &\leq C\epsilon^{-\frac{1}{2}}\|Ext\|\, \|u - P^\partial_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(I)} \end{aligned} \tag{3.98}$$

where $\epsilon > 0$. On the other side, it follows from inequality (3.93) and the inverse inequality for polynomials,

$$|\psi|_{H^{s+\epsilon}} \leq Cp^{2\epsilon}|\psi|_{H^s} \tag{3.99}$$

that,

$$\|\psi\|_{H^{\frac{1}{2}}(I)} \leq \|\psi\|_{H^{\frac{1}{2}+\epsilon}(I)} \leq C\epsilon^{-\frac{1}{2}}\|\psi'\|_{H^{-\frac{1}{2}+\epsilon}(I)} \leq C\epsilon^{-\frac{1}{2}}p^{2\epsilon}\|\psi'\|_{H^{-\frac{1}{2}}(I)} \tag{3.100}$$

Combining (3.98) with (3.100), triangle inequality (3.91), and the projection error estimates, we get,

$$\|u - \Pi^{\partial}_{\frac{1}{2}}u\|_{H^{\frac{1}{2}}(I)} \leq C\epsilon^{-1}p^{3\epsilon}p^{-(r-\frac{1}{2})}\|u\|_{H^r(I)} \tag{3.101}$$

for $\frac{1}{2} + \epsilon \leq r$. Choosing $\epsilon = 1/\ln p$, we have,

$$\ln(p^{\epsilon}) = \epsilon \ln p = 1, \qquad \text{so } p^{\epsilon} = e \tag{3.102}$$

and

$$\epsilon^{-1} = (\ln p)\,. \tag{3.103}$$

We obtain the following result.

THEOREM 3

There exist constants C, independent of p such that,

$$\begin{aligned} \|u - \Pi^{\partial}_{\frac{1}{2}}u\|_{H^{\frac{1}{2}}(I)} &\leq C \ln p\, p^{-(r-\frac{1}{2})}\|u\|_{H^r(I)}, && \forall u \in H^r(I),\\ \|E - \Pi_{-\frac{1}{2}}E\|_{H^{-\frac{1}{2}}(I)} &\leq C \ln(p-1)\,(p-1)^{-(r-\frac{1}{2})}\|E\|_{H^{r-1}(I)}, && \forall E \in H^{r-1}(I), \end{aligned} \tag{3.104}$$

and $\frac{1}{2} < r$. ∎

3.2.3 The case $0 \leq s < \frac{1}{2}$

The interpolation operators are defined on spaces $H^r(I), H^{r-1}(I)$ with $r > \frac{1}{2}$ but the projections are done in the weaker norms. Specifically, we will be interested later in the case $s = 0$, corresponding to interpolation on edges for the 3D case.

$$\begin{array}{ccccccc} \mathbb{R} & \longrightarrow & H^r(I) & \xrightarrow{\partial} & H^{r-1}(I) & \longrightarrow & \{0\} \\ \Big\downarrow & & \Big\downarrow \Pi^{\partial}_s & & \Big\downarrow \Pi_{s-1} & & \Big\downarrow \\ \mathbb{R} & \longrightarrow & \mathcal{P}^p & \xrightarrow{\nabla} & \mathcal{P}^{p-1} & \longrightarrow & \{0\} \end{array} \tag{3.105}$$

The commuting interpolation operators need to be redefined.

$$\begin{cases} \Pi^{\partial}_s u =: u_p \in \mathcal{P}^p(I) \\ u_p = u \text{ at } 0,1 \\ \|u_p - u\|_{H^s(I)} \to \min \end{cases} \tag{3.106}$$

and

$$\left\{\begin{array}{l}\Pi_{s-1}E =: E_p = E_1 + E_{2,p} \in \mathcal{P}^{p-1}(I), \quad E_1 = const,\ E_2 \in \mathcal{P}^{p-1}_{avg}(0,1), \text{and} \\ < E_1 - E, 1 >= 0, \\ \| \int_0^x (E - E_1) - \int_0^x E_{2,p} \|_{H^s(I)} \to \min \end{array}\right. \tag{3.107}$$

In other words, given a distribution E, we first compute its average, then introduce potential,

$$u(x) = \int_0^x (E - E_1) :=< E - E_1, 1_{[0,x]} > , \tag{3.108}$$

project it in the H^s-norm onto polynomials vanishing at the endpoints to get u_p, and differentiate back the projection u_p to get contribution $E_{2,p}$. Notice that, for $s > \frac{1}{2}$ and $u \in H^s(I)$, the projections,

$$\|u - u_p\|_{H^s(I)} \text{ and } \|u' - u'_p\|_{H^{s-1}(I)} \tag{3.109}$$

are equivalent, but with the equivalence constant blowing up for $s = \frac{1}{2}$, due to the breakdown of inequality (3.93).

Exercise 13 Prove that the diagram commutes. ▮

THEOREM 4
Let $0 \le s < r$. There exist constants C, independent of p such that,

$$\begin{array}{lll}\|u - \Pi^{\partial}_s u\|_{H^s(I)} & \le C p^{-(r-s)} \|u\|_{H^r(I)}, & \forall u \in H^r(I), \\ \|E - \Pi_{s-1}E\|_{H^{s-1}(I)} & \le C(p-1)^{-(r-s)} \|E\|_{H^{r-1}(I)}, & \forall E \in H^{r-1}(I)\ . \end{array} \tag{3.110}$$

▮

Proof: We start with the best approximation estimate,

$$\|u - u_p\|_{H^\mu(I)} = \inf_{w_p \in \mathcal{P}^p} \|u - w_p\|_{H^\mu(I)} \le C p^{-(r-\mu)} \|u\|_{H^r(I)} \tag{3.111}$$

Let w be the solution of the dual problem,

$$\left\{\begin{array}{l} w \in H^\mu(I) \\ (\delta u, w)_{H^\mu(I)} = (\delta u, g)_{L^2(I)}, \quad \forall \delta u \in H^\mu(I) \end{array}\right. \tag{3.112}$$

with $g = u - u_p$. We can show, comp. Exercise 14, that,

$$\|w\|_{H^{2\mu}(I)} \le C \|g\|_{L^2(I)} \tag{3.113}$$

Setting $\delta u = u - u_p$, we introduce the best approximation $w_p \in \mathcal{P}^p(I)$ of w in the $H^\mu(I)$-norm, and apply the standard duality argument,

$$
\begin{aligned}
\|u-u_p\|^2_{L^2(I)} &= (u-u_p, w)_{H^\mu(I)} \\
&= (u-u_p, w-w_p)_{H^\mu(I)} \\
&\le \|u-u_p\|_{H^\mu(I)} \|w-w_p\|_{H^\mu(I)} \\
&\le Cp^{-(r-\mu)} \|u\|_{H^r(I)} \, p^{-\mu} \|w\|_{H^{2\mu}(I)} \\
&\le Cp^{-r} \|u\|_{H^r(I)} \|u-u_p\|_{L^2(I)}
\end{aligned}
\tag{3.114}
$$

to conclude that,

$$
\|u-u_p\|_{L^2(I)} \le Cp^{-r} \|u\|_{H^r(I)} \tag{3.115}
$$

Next we define a correction

$$
v(x) = (u(0) - u_p(0))\phi_0(x) + (u(1) - u_p(1))\phi_1(x) \, , \tag{3.116}
$$

where

$$
\begin{cases}
\phi_0 \in \mathcal{P}^p, \quad \phi(0) = 1 \\
\|\phi_0\|_{L^2(I)} \to \min
\end{cases}
\tag{3.117}
$$

with ϕ_1 defined analogously. It has been proved in [34, Lemma 4.1] that

$$
\|v\|_{L^2(I)} \le Cp^{-1} \max\{|u(0) - u_p(0)|, |u(1) - u_p(1)|\} \tag{3.118}
$$

It follows from the Trace Theorem that

$$
\|v\|_{L^2(I)} \le C(\mu)p^{-1} \|u-u_p\|_{H^\mu(I)} \le C(\mu)p^{-1}p^{-(r-\mu)} \|u\|_{H^r(I)} \le Cp^{-r} \|u\|_{H^r(I)} \tag{3.119}
$$

Applying the triangle inequality finishes the argument,

$$
\|u - \Pi_0^\partial u\|_{L^2(I)} \le \|u-u_p\|_{L^2(I)} + \|v\|_{L^2(I)} \le Cp^{-r} \|u\|_{H^r(I)} \tag{3.120}
$$

Now, we can interpolate with s between 0 and any $s > \frac{1}{2}$ to conclude that,

$$
\|u - \Pi_s^\partial u\|_{H^s(I)} \le Cp^{-(r-s)} \|u\|_{H^r(I)} \tag{3.121}
$$

The corresponding estimate for the Π_{s-1} follows now from the commutativity argument, see proof of Theorem 2. ▮

Remark 2 An alternative strategy would be to keep the $H^{\frac{1}{2}}$ interpolation operators for all values of s. The duality argument implies then the optimal p error estimates as well. Thus, in one space dimension, we have at least two alternative families of commuting interpolation operators that yield optimal p-estimates. ▮

Exercise 14 Prove the regularity result (3.113). *Hint:* Use the Fourier series representation of fractional spaces. ▮

3.3 Localization argument

In the next sections, we will need to estimate the 1D interpolation errors over the boundary of a 2D element[6]. We will need the following fundamental result.

Proposition 3
Let $I = (-1, 1),\ I_1 = (-1, 0),\ I_2 = (0, 1)$. *Let* $0 \le s \le 1,\ s \ne \frac{1}{2}$. *There exists a constant* $C > 0$ *such that,*

$$\|u\|_{H^s(I)} \le C \left(\|u\|_{H^s(I_1)} + \|u\|_{H^s(I_2)} \right) \ , \forall u \in H^s(I) \tag{3.122}$$

Here, $u|_{I_i} \in H^s(I_i),\ i = 1, 2$, *and by* $\|u\|_{H^s(I_i)}$ *we understand the norm of the restriction of function* u *in* $H^s(I_i)$. *Moreover, for* $s = \frac{1}{2} + \epsilon$, *or* $s = \frac{1}{2} - \epsilon$, $C = O(\epsilon^{-1})$. ▮

Proof: See [25, p.29-30] or [16]. ▮

The result enables estimating the interpolation error on the boundary of a 2D element. Let $\partial\Omega$ denote the boundary of a 2D polygon Ω, composed of edges e. Let $0 \le s \le 1, s \ne \frac{1}{2}, \quad r > s$. We have

$$\begin{aligned}
\|u - \Pi^{\partial}_s u\|_{H^s(\partial\Omega)} &\le C \sum_e \|u - \Pi^{\partial}_s u\|_{H^s(e)} \\
&\le C p^{-(r-s)} \sum_e \|u\|_{H^r(e)} \\
&\le C p^{-(r-s)} \|u\|_{H^r(\partial\Omega)}
\end{aligned} \tag{3.123}$$

Here p denotes the order of approximation on the element boundary. For an element of variable order, p is the minimum order for all edges, $p = \min_e p_e$. For $s = \frac{1}{2}$, we need to utilize the information about the blow up of constant $C = c(s)$ with $s \to \frac{1}{2}$. We have,

$$\|u - \Pi^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}}(\partial\Omega)} \le \|u - \Pi^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(\partial\Omega)} \le C\epsilon^{-1} \sum_e \|u - \Pi^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(e)} \tag{3.124}$$

But,

$$\begin{aligned}
\|u - \Pi^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(e)} &\le \|u - P^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(e)} + \|P^{\partial}_{\frac{1}{2}} u - \Pi^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(e)} \\
&\le \|u - P^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(e)} + C p^{2\epsilon} \|P^{\partial}_{\frac{1}{2}} u - \Pi^{\partial}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}}(e)} \qquad \text{(inverse inequality)}
\end{aligned} \tag{3.125}$$

The second term is then estimated in the same way as the in Section 3.2.2 resulting in an extra ϵ^{-1} blow-up factor. The extra epsilon in the norm of the projection error present in the first term can be eliminated by

[6] Analogously, for 1D elliptic problems, we estimate the interpolation error over the entire finite element mesh.

using the inverse inequality argument,

$$
\begin{aligned}
\|u-P^{\partial}_{\frac{1}{2}}u\|_{H^{\frac{1}{2}+\epsilon}(e)} &\le \|u-P^{\partial}_{\frac{1}{2}+\epsilon}u\|_{H^{\frac{1}{2}+\epsilon}(e)} + Cp^{2\epsilon}\|P^{\partial}_{\frac{1}{2}+\epsilon}u-P^{\partial}_{\frac{1}{2}}u\|_{H^{\frac{1}{2}}(e)} \\
&\le \|u-P^{\partial}_{\frac{1}{2}+\epsilon}u\|_{H^{\frac{1}{2}+\epsilon}(e)} + Cp^{2\epsilon}\left(\|P^{\partial}_{\frac{1}{2}+\epsilon}u-u\|_{H^{\frac{1}{2}+\epsilon}(e)} + \|u-P^{\partial}_{\frac{1}{2}}u\|)_{H^{\frac{1}{2}}(e)}\right) \\
&\le Cp^{3\epsilon}p^{-(r-\frac{1}{2})}\|u\|_{H^r(e)}
\end{aligned}
\tag{3.126}
$$

for $r > \frac{1}{2}$. The final estimate reads as follows.

$$
\begin{aligned}
\|u-\Pi^{\partial}_{\frac{1}{2}}u\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le C\epsilon^{-2}p^{3\epsilon}p^{-(r-\frac{1}{2})}\sum_e \|u\|_{H^r(e)} \\
&\le C(\ln p)^2 p^{-(r-\frac{1}{2})}\sum_e \|u\|_{H^r(e)} \\
&\le C(\ln p)^2 p^{-(r-\frac{1}{2})}\|u\|_{H^r(\partial\Omega)}
\end{aligned}
\tag{3.127}
$$

We also get the estimate in the negative norms. Let $0 \le s \le 1, s \ne \frac{1}{2}, \quad r > s, r > \frac{1}{2}$, and let $E \in H^{r-1}(\partial\Omega)$. Let E_0 denote the average value of E, i.e.,

$$
< E - E_0, 1 >= 0, \quad \|E_0\|_{H^{r-1}(\partial\Omega)} \le C\|E\|_{H^{r-1}(\partial\Omega)}
\tag{3.128}
$$

Exercise 15 Prove that, for the closed curve $\partial\Omega$, $1 \in H^r(\partial\Omega)$, and the estimate (3.128) holds for any $0 \le r \le 1$. ∎

Notice also that, for a closed curve, the range of the (tangential) derivative coincides with distributions with zero average. Consequently, there exists a potential $u \in H^r(\partial\Omega)$ such that $u' = E - E_0$ and, by the commutativity property,

$$
\left(\Pi^{\partial}_s u|_e\right)' = \Pi_{s-1}u'|_e = \Pi_{s-1}E|_e - E_0
\tag{3.129}
$$

We have,

$$
E - \Pi_{s-1}E = E - \left[\left(\Pi^{\partial}_s u|_e\right)' + E_0\right] = \left(u - \Pi^{\partial}_s u\right)'
\tag{3.130}
$$

and,

$$
\|E-\Pi_{s-1}E\|_{H^{s-1}(\partial\Omega)} \le C\|u-\Pi^{\partial}_s u\|_{H^s(\partial\Omega)} \le C\sum_e \|u-\Pi^{\partial}_s u\|_{H^s(e)}
\tag{3.131}
$$

Let u_0 denote the average value of potential u on edge e. The interpolation operator reproduces constants and this implies that,

$$
\begin{aligned}
\|u-\Pi^{\partial}_s u\|_{H^s(e)} &= \|u-u_0-\Pi^{\partial}_s(u-u_0)\|_{H^s(e)} \\
&\le Cp^{-(r-s)}\|u-u_0\|_{H^r(e)} \\
&\le Cp^{-(r-s)}\|u'\|_{H^{r-1}(e)} \\
&\le Cp^{-(r-s)}\|E-E_0\|_{H^{r-1}(e)}
\end{aligned}
\tag{3.132}
$$

Recalling (3.128), we get,

$$\|E - \Pi_{s-1}E\|_{H^{s-1}(\partial\Omega)} \leq Cp^{-(r-s)}\|E - E_0\|_{H^{r-1}(\partial\Omega)} \leq Cp^{-(r-s)}\|E\|_{H^{r-1}(\partial\Omega)} \tag{3.133}$$

Exercise 16 Let $r > \frac{1}{2}$. Prove that,

$$\|E - \Pi_{-\frac{1}{2}}E\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C(\ln p)^2 p^{-(r-s)}\|E\|_{H^{r-1}(\partial\Omega)} \tag{3.134}$$

∎

Notice that, at this point, we have not claimed any localization results in the negative norms. We do have, however,

Proposition 4
Let $I = (-1,1),\ I_1 = (-1,0),\ I_2 = (0,1)$. Let $0 \leq t < \frac{1}{2}$. There exists a constant $C > 0$ such that,

$$\|E\|_{H^{-t}(I)} \leq C\left(\|E\|_{H^{-t}(I_1)} + \|E\|_{H^{-t}(I_2)}\right) \quad \forall E \in H^{-t}(I) \tag{3.135}$$

Moreover, for $s = \frac{1}{2} + \epsilon$, or $s = \frac{1}{2} - \epsilon$, $C = O(\epsilon^{-\frac{1}{2}})$. ∎

Proof: For the range $0 \leq t < \frac{1}{2}$,

$$\|\phi\|_{\widetilde{H}^t(I)} \leq C\|\phi\|_{H^t(I)} \tag{3.136}$$

with the equivalence constant $C = O(\epsilon^{\frac{1}{2}})$ for $t = \frac{1}{2} - \epsilon$, see [29, p.105]. By the duality argument,

$$\|E\|_{\widetilde{H}^{-\frac{1}{2}+\epsilon}(I)} \leq C\epsilon^{-\frac{1}{2}}\|E\|_{H^{-\frac{1}{2}+\epsilon}(I)} \tag{3.137}$$

Let $\phi \in \mathcal{D}(I)$ be an arbitrary test function. Choose $\phi_n^i \in \mathcal{D}(I_i)$ converging to restriction $\phi|_{I_i}$ in $H^t(I_i)$-norm. Then,

$$| < E, \phi > | \leq \sum_{i=1}^{2} \|E\|_{\widetilde{H}^{-t}(I_i)}\|\phi_n^i\|_{H^t(I_i)} + \|E\|_{\widetilde{H}^{-t}(I)} \sum_{i=1}^{2} \|\phi|_{I_i} - \phi_n^i\|_{H^t(I_i)} \tag{3.138}$$

Passing to the limit with $n \to \infty$, we get,

$$| < E, \phi > | \leq C\left(\sum_{i=1}^{2} \|E\|_{\widetilde{H}^{-t}(I_i)}\right)\|\phi\|_{H^t(I)} \tag{3.139}$$

which, combined with (3.137), finishes the argument. ∎

The localization result allows for an alternative proof of estimate of the interpolation error in the negative norm (3.133) for $s > \frac{1}{2}$. Localization in the dual norm H^{-t} for $t = \frac{1}{2}$ is impossible, but the information about the blow up rate can again be translated into the estimate with the logarithmic term. The same argument can be used in the proof utilizing the commutativity argument. We get,

$$\|E - \Pi_{s-1}E\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C(\ln p)^2 p^{-(r-\frac{1}{2})}\|E\|_{H^{r-1}(\partial\Omega)} \tag{3.140}$$

Localization in the dual norm H^{-t} for $\frac{1}{2} < t \leq 1$ requires extra compatibility conditions for the functional to be localized. This can be immediately seen by considering the delta functional,

$$< \delta, \phi >:= \phi(0) \tag{3.141}$$

Obviously, the delta functional cannot be localized. But we have, for instance,

Exercise 17 Let $I = (-1,1),\ I_1 = (-1,0),\ I_2 = (0,1)$. Let $\frac{1}{2} < t \leq 1$. Let $\phi_0 \in \widetilde{H}^t(I)$ be a specific function such that $\phi_0(0) = 1$. Let $E \in H^{-t}(I)$ be an arbitrary functional such that $< E, \phi_0 >= 0$. Prove that there exists a constant $C > 0$ such that,

$$\|E\|_{H^{-t}(I)} \leq C\left(\|E\|_{H^{-t}(I_1)} + \|E\|_{H^{-t}(I_2)}\right)\ , \tag{3.142}$$

where the constant C depends upon test function ϕ_0 but it is independent of E. Conversely, prove that if, for a particular E, the estimate above holds, then there must exist a function $\phi_0 \in \widetilde{H}^t(I), \phi_0(0) = 1$ such that $< E, \phi_0 >= 0$. ∎

Another sufficient condition can be extracted from the reasoning leading to the estimate (3.133).

Exercise 18 Let $I = (-1,1),\ I_1 = (-1,0),\ I_2 = (0,1)$. Let $\frac{1}{2} < t \leq 1$. Let $E \in H^{-t}(I)$ be such that there exists a potential $u \in H^{1-t}(I),\ u' = E$, such that,

$$\int_{I_i} u = 0, \quad i = 1,2 \tag{3.143}$$

Then (3.142) holds. ∎

4 Commuting Projections and Projection-Based Interpolation Operators in Two Space Dimensions

4.1 Definitions and commutativity

We shall consider the following diagram.

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^r(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}^{r-1}(\text{curl},\Omega) & \xrightarrow{\text{curl}} & H^{r-1}(\Omega) & \longrightarrow & \{0\} \\
\downarrow & & P_s^{grad}\Big\downarrow \Pi_s^{grad} & & P_{s-1}^{curl}\Big\downarrow \Pi_{s-1}^{curl} & & P_{s-1}\Big\downarrow \Pi_{s-1} & & \downarrow \\
\mathbb{R} & \longrightarrow & W_p & \xrightarrow{\nabla} & \boldsymbol{Q}_p & \xrightarrow{\text{curl}} & Y_p & \longrightarrow & \{0\}
\end{array} \tag{4.144}$$

Here $\frac{1}{2} \leq s \leq 1$ with $s \leq r$, $r > 1$, and curl denotes the scalar-valued curl operator in 2D. By $\boldsymbol{H}^{r-1}(\mathrm{curl}, \Omega)$ we understand the space of all vector-valued functions in $\boldsymbol{H}^{r-1}(\Omega)$ whose curl is in $H^{r-1}(\Omega)$. Ω stands for a 2D element, either a quad or a triangle, and $Q_p, \boldsymbol{W}_p, Y_p$ denote any of the exact polynomial sequences defined on element Ω, discussed in Section 2. The common property of those sequences is that the corresponding trace spaces for $Q_p, \boldsymbol{W}_p$ corresponding to any edge e, define the 1D exact polynomial sequence discussed in the previous section.

The projection operators $P_s^{grad}, P_{s-1}^{curl}$ are defined as follows.

$$\begin{cases} P_s^{grad} u =: u_p \in W_p \\ \|\nabla u_p - \nabla u\|_{H^{s-1}(\Omega)} \to \min \\ (u_p - u, 1)_{H^s(\Omega)} = 0 \end{cases} \tag{4.145}$$

$$\begin{cases} P_{s-1}^{curl} \boldsymbol{E} =: \boldsymbol{E}_p \in \boldsymbol{Q}_p \\ \|\mathrm{curl}\boldsymbol{E}_p - \mathrm{curl}\boldsymbol{E}\|_{H^{s-1}(\Omega)} \to \min \\ (\boldsymbol{E}_p - \boldsymbol{E}, \nabla\phi)_{H^{s-1}(\Omega)} = 0,\ \forall \phi \in W_p \end{cases} \tag{4.146}$$

and P_{s-1} denotes the orthogonal projection onto Y_p in the H^{s-1}-norm.

Exercise 19 Show that the projections defined above make the diagram (4.144) commute. ∎

The projection-based interpolation operators are defined as follows.

$$\begin{cases} \Pi_s^{grad} u =: u_p \in W_p \\ u_p = \Pi_{s-\frac{1}{2}}^{\partial} u \text{ on } \partial\Omega \\ \|\nabla u_p - \nabla u\|_{H^{s-1}(\Omega)} \to \min \end{cases} \tag{4.147}$$

$$\begin{cases} \Pi_{s-1}^{curl} \boldsymbol{E} =: \boldsymbol{E}_p \in \boldsymbol{Q}_p \\ E_{t,p} = \Pi_{s-\frac{3}{2}} E_t \text{ on } \partial\Omega \\ \|\mathrm{curl}\boldsymbol{E}_p - \mathrm{curl}\boldsymbol{E}\|_{H^{s-1}(\Omega)} \to \min \\ (\boldsymbol{E}_p - \boldsymbol{E}, \nabla\phi)_{H^{s-1}(\Omega)} = 0,\ \forall \phi \in W_p \ :\ \phi = 0 \text{ on } \partial\Omega \end{cases} \tag{4.148}$$

and

$$\begin{cases} \Pi_{s-1} v =: v_p \in Y_p \\ < v_p - v, 1 >= 0 \\ \|v_p - v\|_{H^{s-1}(\Omega)} \to \min \end{cases} \tag{4.149}$$

Here Π_s^{∂}, Π_s are the 1D interpolation operators discussed in the previous section, and $E_t, E_{t,p}$ denote the tangential component of $\boldsymbol{E}, \boldsymbol{E}_p$, respectively. Notice that all minimization problems are *constrained minimization problems* - the boundary values of the interpolants in (4.147),(4.148), and the average value of the interpolant in (4.149), are fixed. Similarly to 1D, the projection operators can be interpreted as local minimization problems with Neumann boundary conditions, while the interpolation operators employ local Dirichlet boundary conditions. Finally, remember that by the boundary values of fields $\boldsymbol{E} \in \boldsymbol{H}^{r-1}(\mathrm{curl}, T)$, we always understand the trace of the tangential component E_t. Definition of the tangential component E_t is non-trivial. For $\boldsymbol{E} \in \boldsymbol{H}^r, \frac{1}{2} < r < \frac{3}{2}$, the tangential component is understood in the sense of the trace theorem and we have, comp. [29, p.102],

$$\|\boldsymbol{E}_t\|_{H^{r-\frac{1}{2}}(\partial\Omega)} \leq C\|\boldsymbol{E}\|_{H^r(\Omega)} \tag{4.150}$$

The definition for the range $-\frac{1}{2} < r < \frac{1}{2}$ is more complicated. We consider first a sufficiently regular field $\boldsymbol{E} \in \boldsymbol{H}^r(\Omega)$ and a test function $\phi \in H^{\frac{1}{2}-r}(\partial\Omega)$, to invoke the integration by parts formula:

$$\int_{\partial\Omega} E_t\phi = \int_{\Omega}(\mathrm{curl}\boldsymbol{E})\Phi - \int_{\Omega}\boldsymbol{E}(\nabla\times\Phi) \tag{4.151}$$

Here $\Phi \in H^{1-r}(\Omega)$ is an extension of ϕ such that,

$$\|\Phi\|_{H^{1-r}(\Omega)} \leq C\|\phi\|_{H^{\frac{1}{2}-r}(\partial\Omega)} \tag{4.152}$$

see [29, p.101], and $\nabla\times$ denotes the vector-valued curl operator in 2D. We have,

$$\begin{aligned}
|\int_{\partial\Omega} E_t\phi| &\leq \|\mathrm{curl}\boldsymbol{E}\|_{H^s(\Omega)}\|\Phi\|_{\widetilde{H}^{-s}(\Omega)} + \|\boldsymbol{E}\|_{H^r(\Omega)}\|\nabla\times\Phi\|_{\widetilde{H}^{-r}(\Omega)} \\
&\leq C\left(\|\mathrm{curl}\boldsymbol{E}\|_{H^s(\Omega)}\|\Phi\|_{H^{1-r}(\Omega)} + \|\boldsymbol{E}\|_{H^r(\Omega)}\|\nabla\times\Phi\|_{H^{-r}(\Omega)}\right) \\
&\leq C\left(\|\mathrm{curl}\boldsymbol{E}\|_{H^s(\Omega)} + \|\boldsymbol{E}\|_{H^r(\Omega)}\right)\|\phi\|_{H^{\frac{1}{2}-r}(\partial\Omega)}
\end{aligned} \tag{4.153}$$

where the range of r secures the equivalence of $\widetilde{H}^r$- and H^r-norms, and $-\frac{1}{2} < s < \frac{1}{2}$ is an arbitrary, not necessarily related to r parameter (we may, of course, choose $s = r$). The range of s implies that $-s \leq 1-r$. The density argument allows now to extend the notion of the tangential component for every field $\boldsymbol{E} \in \boldsymbol{H}^r$ such that $\mathrm{curl}\boldsymbol{E} \in H^s(\Omega)$, with both r and s from interval $(-\frac{1}{2}, \frac{1}{2})$. We get the estimate,

$$\|E_t\|_{H^{r-\frac{1}{2}}(\partial\Omega)} \leq C\left(\|\mathrm{curl}\boldsymbol{E}\|_{H^s(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}^r(\Omega)}\right) \tag{4.154}$$

which can be seen as an equivalent of the Trace Theorem for the space,

$$\{\boldsymbol{E} \in \boldsymbol{H}^r(\Omega) \;:\; \mathrm{curl}\,\boldsymbol{E} \in H^s(\Omega)\} \tag{4.155}$$

Note that the blow up of the equivalence constants prohibits extending the definition to values $s, r = -\frac{1}{2}, \frac{1}{2}$, and that in both cases the constants are of order $O(\epsilon^{\frac{1}{2}})$ for $s, r = -\frac{1}{2} + \epsilon$ or $s, r = \frac{1}{2} - \epsilon$.

Exercise 20 Show that the definition of E_t discussed above is independent of the choice of extension Φ. ▮

THEOREM 5

The interpolation operators make the diagram (4.144) commute. ∎

Proof: The comuttativity of the first block follows from the fact that operator Π_s^{grad} preserves constants. In order to show the commutativity of the second block, we need to demonstrate that,

$$\Pi_{s-1}^{curl} \nabla u = \nabla \Pi_s^{grad} u \tag{4.156}$$

Let $\boldsymbol{E} = \nabla u$. By the commutativity of the 1D diagram, we have,

$$E_{t,p} = \Pi_{s-\frac{3}{2}} \frac{\partial u}{\partial t} = \frac{\partial u_p}{\partial t} \tag{4.157}$$

where $u_p = \Pi_{s-\frac{1}{2}}^{\partial} u$ and $\frac{\partial}{\partial t}$ denotes the tangential derivative on the boundary of the element. Consequently,

$$\int_\Omega \mathrm{curl} \boldsymbol{E}_p = \int_{\partial\Omega} E_{t,p} = \int_{\partial\Omega} \frac{\partial u_p}{\partial t} = 0 \tag{4.158}$$

At the same time,

$$(\mathrm{curl} \boldsymbol{E}_p, \mathrm{curl} \boldsymbol{F})_{H^{s-1}(\Omega)} = 0 \tag{4.159}$$

for every $\boldsymbol{F} \in \boldsymbol{Q}_p, F_t = 0$ on $\partial\Omega$. However, the image of such polynomials F coincides exactly with polynomials in Y_p with zero average, where the curl of $\boldsymbol{E}_p$ lives. Consequently, $\mathrm{curl}\boldsymbol{E}_p = 0$ and $E_p = \nabla u_p$ for some $u_p \in W_p$. Substituting ∇u_p into (4.148)$_4$ we learn that $u_p = \Pi_s^{grad} u$.

To prove the last commutativity property, we need to show that,

$$\Pi_{s-1}(\mathrm{curl} \boldsymbol{E}) = \mathrm{curl}\, (\Pi_{s-1}^{curl} \boldsymbol{E}) \tag{4.160}$$

Let $\boldsymbol{E}_p = \Pi_{s-1}^{curl} \boldsymbol{E}$. It follows form the definition of the 1D interpolation operator that,

$$< \mathrm{curl} \boldsymbol{E}_p - \mathrm{curl} \boldsymbol{E}, 1 > = < E_{t,p} - E_t, 1 > = \ 0 \tag{4.161}$$

Finally, condition (4.149)$_3$ follows directly from condition (4.148)$_3$. ∎

4.2 Polynomial preserving extension operators

Let u_p be the trace of a polynomial from space W_p defined on the boundary of the element. Of fundamental importance for the presented theory is the existence of a polynomial extension $U_p \in W_p,\ U_p|_{\partial\Omega} = u_p$ such that,

$$\|U_p\|_{H^s(\Omega)} \le C \|u_p\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \tag{4.162}$$

with constant C *independent of* p. Here, we are interested in the range $\frac{1}{2} \le s \le 1$. A more demanding request is to look for a general extension operator,

$$\mathrm{Ext}\ :\ H^{s-\frac{1}{2}}(\partial\Omega) \ni u \to U \in H^s(\Omega) \tag{4.163}$$

that is continuous and polynomial preserving. For $s = 1$ and both triangular and rectangular elements, the operator of this type was first constructed by Babuška and Suri in [4], see also Babuška et al. in [3]. For a triangular element, different, explicit constructions were shown by Munoz-Sola in [30], Ainsworth and Demkowicz in [1] and Schoeberl et al. in [36]. In particular, the explicit construction in [1] allows for an immediate proof of continuity for fractional norms. For a square element, one can also use discrete harmonic extensions studied by Pavarino and Widlund in [34]. To conclude the independence of constant C in (4.162) of p, one has then to use the results of Maday [28], see [34, p.1316]. All these results are quite technical.

In two space dimensions, the existence of a polynomial extension for the H^s-space, implies immediately an analogous result for the $\boldsymbol{H}^{s-1}(\mathrm{curl})$ space. Indeed, let $E_{t,p}$ be the tangential trace of a polynomial in $\boldsymbol{Q}_p$. Let E_0 be the average value of $E_{t,p}$ on the boundary of the element. By the result of Exercise 15, the average E_0 depends continuously upon the $H^{s-\frac{3}{2}}$-norm of $E_{t,p}$. Let $u \in H^{s-\frac{1}{2}}(\partial\Omega)$ be a polynomial of zero average such that $u' = E_{t,p} - E_0$. Let U be then the polynomial extension of u discussed above, and let $\boldsymbol{E}_0$ be the lowest order extension of the constant average E_0. Define $Ext^{curl} E_{t,p} = \boldsymbol{E}_p := \nabla U_p + \boldsymbol{E}_0$. We have,

$$
\begin{aligned}
\|\boldsymbol{E}_p\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} &\leq \|\nabla U_p\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} + \|\boldsymbol{E}_0\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} \\
&\leq C\left(\|U_p\|_{H^s(\Omega)} + |E_0|\right) \\
&\leq C\left(\|u_p\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + |E_0|\right) \\
&\leq C\left(\|E_{t,p} - E_0\|_{H^{s-\frac{3}{2}}(\partial\Omega)} + |E_0|\right) \\
&\leq C\,\|E_{t,p}\|_{H^{s-\frac{3}{2}}(\partial\Omega)}
\end{aligned}
\tag{4.164}
$$

4.3 Right-inverse of the curl operator. Discrete Friedrichs Inequality

Let Ω denote the master triangle or rectangle. Recall the operator K introduced in the discussion of Nédélec's triangle of the first type,

$$
K\psi(\boldsymbol{x}) = -\boldsymbol{x} \times \left(\int_0^1 t\psi(t\boldsymbol{x})\,dt\right)\,\boldsymbol{e}_3\,,
\tag{4.165}
$$

where $\boldsymbol{x} = (x_1, x_2, 0), \boldsymbol{e}_3 = (0, 0, 1)$.

Exercise 21 Prove that the operator K maps space Y_p into the space $\boldsymbol{Q}_p$ for all three Nédélec elements: the triangles of the first and second type, as well as the rectangle of the first type. Verify that,

$$
\mathrm{curl}(K\psi) = \psi
\tag{4.166}
$$

▮

We will show now that operator K is a continuous operator from $H^{-s}(\Omega)$ into $\boldsymbol{H}^{-s}(\mathrm{curl}, \Omega)$. We are interested in the range $0 \leq s \leq \frac{1}{2}$. In order to demonstrate the continuity in the negative exponent norm, we compute first the adjoint operator. Switching to polar coordinates (r, θ), we obtain,

$$K\psi(r,\theta) = r\int_0^1 t\psi(tr,\theta)\,dt\,\boldsymbol{e}_\theta = \frac{1}{r}\int_0^r s\psi(s,\theta)\,ds\,\boldsymbol{e}_\theta\;,$$

where $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ denote the unit vectors of the polar coordinate system. Representing the argument of the dual operator in the polar coordinates as $\boldsymbol{\phi} = \phi_r\boldsymbol{e}_r + \phi_\theta\boldsymbol{e}_\theta$, we get,

$$\begin{aligned}
\int_f K\psi\boldsymbol{\phi}\,d\boldsymbol{x} &= \int_0^{\frac{\pi}{2}}\int_0^{\hat{r}(\theta)}\frac{1}{r}\int_0^r s\psi(s,\theta)\,ds\phi_\theta\,rdrd\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\hat{r}(\theta)} s\psi(s,\theta)\int_s^{\hat{r}(\theta)}\phi_\theta(r,\theta)\,drdsd\theta \\
&= \int_f \psi \underbrace{\int_s^{\hat{r}(\theta)}\phi_\theta(r,\theta)\,dr}_{K^*\boldsymbol{\phi}(s,\theta)}\;d\boldsymbol{x}\;.
\end{aligned}$$

Symbol $\hat{r}(\theta)$ is explained in Fig 1.

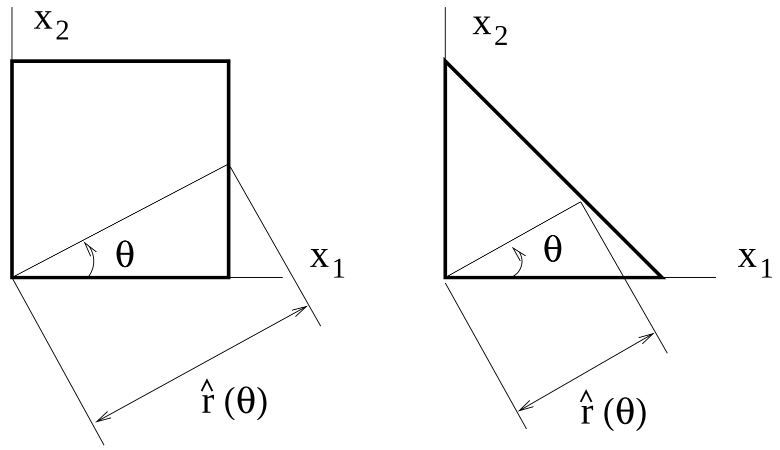

Figure 1: Polar coordinates and integration over the master element

A “brute force” estimate follows.

$$
\begin{aligned}
\|K^*\boldsymbol{\phi}\|^2_{0,f} &= \int_0^{\frac{\pi}{2}}\int_0^{\hat r(\theta)}\left(\int_r^{\hat r(\theta)}\phi_\theta(s,\theta)\,ds\right)^2 rdrd\theta \\
&\le \int_0^{\frac{\pi}{2}}\int_0^{\hat r(\theta)}\underbrace{\int_r^{\hat r(\theta)} s^{-\frac{1}{2}}\,ds}_{2(\sqrt{\hat r(\theta)}-\sqrt{r})\le 2}\int_r^{\hat r(\theta)} s\phi_\theta^2(s,\theta)\,ds\,rdrd\theta \\
&\le 2\int_0^{\frac{\pi}{2}}\underbrace{\int_0^{\hat r(\theta)} r\,dr}_{\le\frac{1}{2}}\int_0^{\hat r(\theta)} s\phi_\theta^2(s,\theta)\,dsd\theta \\
&\le \|\phi_\theta\|^2_{L^2(\Omega)}\,.
\end{aligned}
$$

Operator K^* is thus a continuous operator from $\boldsymbol{L}^2(\Omega)$ into $L^2(\Omega)$. We compute now the gradient,

$$
\nabla K^*\boldsymbol{\phi}(r,\theta) \;= -\phi_\theta(r,\theta)\,\boldsymbol{e}_r + \frac{1}{r}\left(\underbrace{\phi_\theta(\hat r(\theta),\theta)}_{=0}\frac{d\hat r}{d\theta}(\theta) + \int_r^{\hat r(\theta)}\frac{\partial\phi_\theta}{\partial\theta}(s,\theta)\,ds\right)\boldsymbol{e}_\theta\,,
$$

where we have assumed that $\boldsymbol{\phi}$ vanishes on the boundary. The first term estimates trivially, and for the second we have,

$$
\begin{aligned}
\|\frac{1}{r}\int_r^{\hat r(\theta)}\frac{\partial\phi_\theta}{\partial\theta}(s,\theta)\,ds\|^2_{L^2(\Omega)} &= \int_0^{\frac{\pi}{2}}\int_0^{\hat r(\theta)}\frac{1}{r^2}\left(\int_r^{\hat r(\theta)}\frac{\partial\phi_\theta}{\partial\theta}(s,\theta)\,ds\right)^2 rdrd\theta \\
&\le \int_0^{\frac{\pi}{2}}\int_0^{\hat r(\theta)}\frac{1}{r}\underbrace{\int_r^{\hat r(\theta)} 1\,ds}_{\le\sqrt{2}}\int_r^{\hat r(\theta)}(\frac{\partial\phi_\theta}{\partial\theta})^2(s,\theta)\,ds\,drd\theta \\
&\le \sqrt{2}\int_0^{\frac{\pi}{2}}\int_0^{\hat r(\theta)}\int_r^{\hat r(\theta)}\frac{1}{s}(\frac{\partial\phi_\theta}{\partial\theta})^2(s,\theta)\,ds\,drd\theta \\
&\le \sqrt{2}\,\|\frac{1}{r}\frac{\partial\phi_\theta}{\partial\theta}\|^2_{L^2(\Omega)}\,.
\end{aligned}
$$

Consequently, operator K^* is also continuous from $\boldsymbol{H}^1_0(\Omega)$ into $H^1_0(\Omega)$. By the standard interpolation argument, see [29, p.330], operator K^* is continuous from $\widetilde{\boldsymbol{H}}^s(\Omega)$ into $\widetilde{H}^s(\Omega)$ and, consequently, operator K is continuous from $H^{-s}(\Omega)$ into $\boldsymbol{H}^{-s}(\Omega)$.

For elements of variable order, the operator (4.165) still has to be modified to have a range in the right polynomial space. The issue is with the polynomial degree on the boundary. For $\psi \in Y_p$, the tangential trace of $K\psi$ vanishes on edges $x_1 = 0, x_2 = 0$ but it has, in general, a non-zero tangential trace on the rest of the boundary. We utilize the extension operator discussed above and define the ultimate operator as follows,

$$
K^{mod}\psi = (I - Ext^{curl}Tr)K\psi_0 + \boldsymbol{E}_0\,. \tag{4.167}
$$

Here $\psi = \psi_0 + c$ is the decomposition of function ψ into a constant c, and a function ψ_0 with zero average. If all edge orders $p_e = -1$, then $c = 0$. Otherwise $\boldsymbol{E}_0$ denotes any linear combination of lowest order shape functions in $\boldsymbol{Q}_p$ whose curl reproduces the constant c. Notice that, due to the commuting diagram property, K^{mod} is still a right inverse of the curl operator. For $\psi \in Y_p$ with zero average, $K^{mod}\psi$ has a zero tangential trace.

Lemma 2

(Discrete Friedrichs Inequality for fractional spaces in 2D)

Let $0 \leq s \leq \frac{1}{2}$. *There exist* $C > 0$ *such that,*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}^{-s}(\Omega)} \leq C \| \operatorname{curl} \boldsymbol{E} \|_{H^{-s}(\Omega)} , \tag{4.168}$$

for every discrete divergence free polynomial $\boldsymbol{E} \in \boldsymbol{Q}_p$, *i.e.,*

$$(\boldsymbol{E}, \nabla \phi)_{H^{-s}(\Omega)} = 0, \quad \forall \phi \in W_p \, . \tag{4.169}$$

∎

Note that the result covers the case of polynomials with zero tangential trace.

Proof: We utilize the properties of the right-inverse K of the curl operator.

$$\begin{aligned} \|\boldsymbol{E}\|_{\boldsymbol{H}^{-s}(\Omega)} &= \inf_{\phi \in W_p} \|\boldsymbol{E} - \nabla \phi\|_{\boldsymbol{H}^{-s}(\Omega)} \\ &\leq \|\boldsymbol{E} - (\boldsymbol{E} - K^{mod}(\nabla \times \boldsymbol{E}))\|_{\boldsymbol{H}^{-s}(\Omega)} \\ &\leq \|K^{mod}\| \, \|\nabla \times \boldsymbol{E}\|_{\boldsymbol{H}^{-s}(\Omega)} \, . \end{aligned} \tag{4.170}$$

∎

We shall also need a generalization of the classical Poincare inequalities to the case of fractional Sobolev spaces.

Lemma 3

(Poincare's inequalities for fractional spaces in 2D)

Let $\frac{1}{2} < s \leq 1$. *There exist* $C > 0$ *such that*

$$\|u\|_{L^2(\Omega)} \leq C |u|_{H^s(\Omega)} \approx C \|\nabla u\|_{\boldsymbol{H}^{s-1}(\Omega)} , \tag{4.171}$$

for every function $u \in H^s(\Omega)$ *belonging to either of the two families:*

Case 1: $< u, 1 >= 0$,

Case 2: $u = 0$ *on* $\partial\Omega$. ∎

Proof: First of all, it is easy to see that $|.|_{H^s(\Omega)}$ is a norm on the subspaces of $H^s(\Omega)$ corresponding to Case 1 and Case 2. Indeed, let $u\ :\ |u|_{H^s(\Omega)} = 0$, then $\nabla u = 0$, which implies that u is constant over Ω, thus $u = 0$ in both cases. The result follows now for instance from the compact embedding of both spaces into space $L^2(\Omega)$. Assume, by contradiction, that there exists a sequence of functions u_n such that $\|u_n\|_{0,f} = 1$, and

$$|u_n|_{H^s(\Omega)} \le n^{-1} \,. \tag{4.172}$$

By the compactness argument, we can extract a subsequence, denoted with the same symbol, converging weakly in the seminorm and strongly in the L^2 norm to a limit u. Passing to the limit in the inequality above, we get

$$|u|_{H^s(\Omega)} \le \liminf_{n\to\infty} |u_n|_{H^s(\Omega)} = 0 \,, \tag{4.173}$$

thus, $u = 0$. This contradicts the fact that $\|u\|_{L^2(\Omega)} = 1$. ∎

4.4 Projection error estimates

As in the 1D case, it is illuminating to see that the definitions of commuting projection operators $P_s^{grad}, P_{s-1}^{curl}$ are equivalent to the solution of mixed problems. The mixed formulation for determining $P_s^{grad}u$ looks as follows.

$$\left\{\begin{array}{lll} P_s^{grad}u =: u_p \in W_p,\ \lambda \in \mathbb{R} & & \\ (\nabla u_p - \nabla u, \nabla v)_{H^{s-1}(\Omega)} + (\lambda, v)_{H^s(\Omega)} & = 0 & \forall v \in W_p \\ (u_p - u, \mu)_{H^s(\Omega)} & = 0 & \forall \mu \in \mathbb{R} \end{array}\right. \tag{4.174}$$

Substituting $v = \mathrm{const} = \lambda$ in the first equation, we learn that the Lagrange multiplier $\lambda = 0$. The fact that constants are included in space W_p implies the satisfaction of the first Brezzi's inf-sup condition. The inf-sup in kernel condition is implied by the Poincare's inequality, case 1.

The situation is similar with the mixed formulation for determining $P_{s-1}^{curl}\boldsymbol{E}$.

$$\left\{\begin{array}{lll} P_{s-1}^{curl}\boldsymbol{E} =: \boldsymbol{E}_p \in \boldsymbol{Q}_p,\ \psi \in W_p & & \\ (\mathrm{curl}\,\boldsymbol{E}_p - \mathrm{curl}\,\boldsymbol{F}, \mathrm{curl}\,\boldsymbol{F})_{H^{s-1}(\Omega)} + (\nabla\psi, \boldsymbol{F})_{H^{s-1}(\Omega)} & = 0 & \forall \boldsymbol{F} \in \boldsymbol{Q}_p \\ (\boldsymbol{E}_p - \boldsymbol{E}, \nabla\phi)_{H^{s-1}(\Omega)} & = 0 & \forall \psi \in W_p \end{array}\right. \tag{4.175}$$

Substituting $\boldsymbol{F} = \nabla\psi$ into the first equation, we learn again that $\nabla\psi = 0$. The exact sequence property, i.e. the inclusion $\nabla W_p \subset \boldsymbol{Q}_p$ implies the automatic satisfaction of the first inf-sup condition with constant $\beta = 1$, and the discrete Friedrichs inequality proved in Lemma 2 implies the inf-sup in kernel condition. We can conclude the projection error estimates.

THEOREM 6

(Commuting projection error estimates in 2D)

Let $\frac{1}{2} \leq s \leq 1,\ r > s, r > 1$. There exist constants $C > 0$, independent of p, such that,

$$
\begin{aligned}
\|u - P_s^{grad} u\|_{H^s(\Omega)} &\leq C \inf_{u_p \in W_p} \|u - u_p\|_{H^s(\Omega)} && \leq C p^{-(r-s)} \|u\|_{H^r(\Omega)} \\
\|\boldsymbol{E} - P_{s-1}^{curl} \boldsymbol{E}\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} &\leq C \inf_{\boldsymbol{E}_p \in \boldsymbol{Q}_p} \|\boldsymbol{E} - \boldsymbol{E}_p\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} && \leq C p^{-(r-s)} \|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathrm{curl},\Omega)} \\
\|\boldsymbol{v} - P_{s-1} v\|_{H^{s-1}(\Omega)} &= \inf_{v_p \in Y_p} \|v - v_p\|_{H^{s-1}(\Omega)} && \leq C p^{-(r-s)} \|v\|_{H^{r-1}(\Omega)}
\end{aligned}
\tag{4.176}
$$

for every $u \in H^r(\Omega), \boldsymbol{E} \in \boldsymbol{H}^{r-1}(\mathrm{curl}, \Omega), v \in H^{r-1}(\Omega)$. ∎

Proof: For the best approximation results in the H^s-norm, see [37], and in the $\boldsymbol{H}^{r-1}(\mathrm{curl}, \Omega)$-norm, see [18]. ∎

As in 1D, estimating the projection error with the best approximation error can be done directly by using the right-inverse of the curl operator and an analogous, polynomial preserving, right-inverse of the grad operator,

$$
G\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{x} \cdot \int_0^1 \boldsymbol{E}(t\boldsymbol{x})\, dt \tag{4.177}
$$

where $\boldsymbol{x} = (x_1, x_2)$ and $\cdot$ denotes the dot product.

Exercise 22 Let Ω be the square or triangular master element, and $\frac{1}{2} \leq s \leq 1$. Let $\boldsymbol{E} \in \mathcal{R}(\nabla)$. Prove that:

- $\nabla(G\boldsymbol{E}) = \boldsymbol{E}$,
- $E_t = 0$ on $\partial\Omega$ implies $G\boldsymbol{E} = 0$ on $\partial\Omega$,
- operator (4.177) is a continuous, polynomial preserving operator from $\boldsymbol{H}^{s-1}(\mathrm{curl}, \Omega)$ into $H^s(\Omega)$.

∎

The two commuting projection operators can then be represented in the form [22],

$$
\begin{aligned}
P_{s-1}^{curl} \boldsymbol{E} &= P_{s-1}^{curl0}(\boldsymbol{E} - K P_{s-1}(\mathrm{curl}\, \boldsymbol{E})) + K P_{s-1}(\mathrm{curl}\, \boldsymbol{E}) \\
P_s^{grad} u &= P_s^{grad0}(u - G P_{s-1}^{curl0}(\nabla u)) + G P_{s-1}^{curl0}(\nabla u)
\end{aligned}
\tag{4.178}
$$

where P_{s-1}^{curl0} and P_s^{grad0} denote the orthogonal projections in $\boldsymbol{H}^{s-1}$- and H^s-norms onto the subspaces of polynomials in $\boldsymbol{Q}_p$ and W_p with zero curl and gradient, respectively (i.e. onto the range of the gradient operator and constants). The continuity of the right-inverses, and the polynomial-preserving property imply then that the commuting projection errors are bounded by the best approximation errors.

4.5 Interpolation error estimates

The interpolation error estimates are derived by comparing the interpolation errors with the commuting projection errors following the same procedure as in Section 3.2.

THEOREM 7

(Commuting interpolation error estimates in 2D)

Let $\frac{1}{2} < s \leq 1,\ r > s, r > 1$. *There exist constants* $C > 0$, *independent of* p, *such that,*

$$
\begin{aligned}
\|u - \Pi_s^{grad} u\|_{H^s(\Omega)} &\leq C \left(\|u - P_s^{grad} u\|_{H^s(\Omega)} + \|u - \Pi^{\partial}_{s-\frac{1}{2}} u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \right) \\
\|\boldsymbol{E} - \Pi_{s-1}^{curl} \boldsymbol{E}\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} &\leq C \left(\|\boldsymbol{E} - P_{s-1}^{curl} \boldsymbol{E}\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} + \|E_t - \Pi^{curl}_{s-\frac{3}{2}} E_t\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \right) \\
\|v - \Pi_{s-1} v\|_{H^{s-1}(\Omega)} &\leq C \left(\|v - P_{s-1} v\|_{H^{s-1}(\Omega)} + \|v - \Pi_{s-\frac{3}{2}} v\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \right)
\end{aligned}
\tag{4.179}
$$

for every $u \in H^r(\Omega), \boldsymbol{E} \in \boldsymbol{H}^{r-1}(\mathrm{curl},\Omega), v \in H^{r-1}(\Omega)$. *For* $s = \frac{1}{2} + \epsilon$, *constant* $C = O(\epsilon^{-1})$. *Combined with estimates (3.123), (3.127), and (3.133), (3.134), we obtain the following error estimates for the case* $s = 1$,

$$
\begin{aligned}
\|u - \Pi_1^{grad} u\|_{H^1(\Omega)} &\leq C (\ln p)^2\, p^{-(r-1)} \|u\|_{H^r(\Omega)} \\
\|\boldsymbol{E} - \Pi_{-1}^{curl} \boldsymbol{E}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} &\leq C (\ln p)^2\, p^{-(r-1)} \|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathrm{curl},\Omega)}
\end{aligned}
\tag{4.180}
$$

and case $s = \frac{1}{2}$,

$$
\begin{aligned}
\|u - \Pi_{\frac{1}{2}}^{grad} u\|_{H^{\frac{1}{2}}(\Omega)} &\leq C \ln p\, p^{-(r-\frac{1}{2})} \|u\|_{H^r(\Omega)} \\
\|\boldsymbol{E} - \Pi_{-\frac{1}{2}}^{curl} \boldsymbol{E}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\Omega)} &\leq C \ln p\, p^{-(r-\frac{1}{2})} \|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathrm{curl},\Omega)} \\
\|v - \Pi_{-\frac{1}{2}} v\|_{H^{-\frac{1}{2}}(\Omega)} &\leq C \ln p\, p^{-(r-\frac{1}{2})} \|v\|_{H^{r-1}(\Omega)}
\end{aligned}
\tag{4.181}
$$

∎

Proof: **Operator** Π_s^{grad}. We begin with the triangle inequality,

$$
\|u - \Pi_s^{grad} u\|_{H^s(\Omega)} \leq \|u - P_s^{grad} u\|_{H^s(\Omega)} + \|P_s^{grad} u - \Pi_s^{grad} u\|_{H^s(\Omega)} \tag{4.182}
$$

Polynomial $\psi = P_s^{grad} u - \Pi_s^{grad} u \in W_p$ satisfies,

$$
(\nabla\psi, \nabla\phi)_{H^{s-1}(\Omega)} = 0, \quad \forall \phi \in W_p\ :\ \phi = 0 \text{ on } \partial\Omega \tag{4.183}
$$

and, therefore, it is the discrete minimum energy extension with the energy defined by the $H^{s-1}(\Omega)$-norm of gradient $\nabla\psi$. By the Poincare inequality, case 2,

$$
\|\psi\|_{H^s(\Omega)} \leq C \|\nabla\psi\|_{H^{s-1}(\Omega)} \tag{4.184}
$$

where $C = O(\epsilon^{\frac{1}{2}})$ for $s = \frac{1}{2} + \epsilon$. Denoting the trace of ψ by $tr\psi$, we therefore have,

$$\begin{aligned}
\|\psi\|_{H^s(\Omega)} &\le C\|\nabla(Ext\, tr\psi)\|_{H^{s-1}(\Omega)} \\
&\le C\|Ext\, tr\psi\|_{H^s(\Omega)} \\
&\le C\|Ext\|\, \|\Pi_s^{grad}u - P_s u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\
&\le C\|Ext\| \left(\|u - P_s^{grad}u\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|u - \Pi_s^{grad}u\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\right) \\
&\le C\|Ext\| \left(C_{tr}\|u - P_s^{grad}u\|_{H^s(\Omega)} + \|u - \Pi_{s-\frac{1}{2}}^{\partial}u\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\right)
\end{aligned} \tag{4.185}$$

Here C_{tr} is the norm of the trace operator of order $O(\epsilon^{-\frac{1}{2}})$ for $s = \frac{1}{2} + \epsilon$, comp. [29, p.100]. Combining the triangle inequality (4.182) with the result above, we see that the interpolation error is bounded by the projection error, and the interpolation error on the boundary. The estimate for the case $s = \frac{1}{2}$ follows from the arguments discussed in Section 3.

Operator Π_{s-1}^{curl}. We follow exactly the same arguments as for the first case. If $\boldsymbol{\psi} = \Pi_{s-1}^{curl}\boldsymbol{E} - P_{s-1}^{curl}\boldsymbol{E}$, the discrete Friedrichs inequality corresponding to the subspace,

$$\{\boldsymbol{E} \in \boldsymbol{Q}_p \;:\; E_t = 0 \text{ on } \partial\Omega, \quad (\boldsymbol{E}, \nabla\phi)_{H^{s-1}(\Omega)} = 0,\ \forall\phi \in W_p \;:\; \phi = 0 \text{ on } \partial\Omega\} \tag{4.186}$$

is needed to establish the bound,

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\Omega)} \le C\|\,\mathrm{curl}\,\boldsymbol{\psi}\|_{H^{s-1}(\Omega)} \le C\|\,\mathrm{curl}(Ext^{curl}tr\boldsymbol{\psi})\|_{H^{s-1}(\Omega)} \tag{4.187}$$

and the constant C and trace constant C_{tr} in (4.155) experience the same blow up as in the case of operator Π_s^{grad}.

Operator Π_{s-1}. The proof follows from the commutativity of the operators and the result for operator Π_{s-1}^{curl}. ∎

4.6 Localization argument

Proposition 3 remains true in multiple space dimensions, see [25, p.29-30] or [16], and it makes it possible to generalize the interpolation error estimates to the boundary of a 3D polyhedral domain consisting of triangular or rectangular faces. Let $\partial\Omega$ denote the boundary of a 3D polyhedron Ω, composed of faces f. Let $\frac{1}{2} < s \le 1, \quad r > s$. We have

$$\begin{aligned}
\|u - \Pi_s^{grad}u\|_{H^s(\partial\Omega)} &\le C\sum_f \|u - \Pi_s^{grad}u\|_{H^s(f)} \\
&\le Cp^{-(r-s)}\sum_f \|u\|_{H^r(f)} \\
&\le Cp^{-(r-s)}\|u\|_{H^r(\partial\Omega)}
\end{aligned} \tag{4.188}$$

Here p denotes the order of approximation on the element boundary. For the element of variable order, p is the minimum order for all faces and edges. For $s = \frac{1}{2}$, we need to utilize the information about the blow up of constant $C = C(s)$ with $s \to \frac{1}{2}$. We have for $r > 1$,

$$\begin{aligned}
\|u - \Pi^{grad}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le \|u - \Pi^{grad}_{\frac{1}{2}} u\|_{H^{\frac{1}{2}+\epsilon}(\partial\Omega)} \\
&\le C\epsilon^{-1} \sum_f \|u - \Pi^{grad}_{\frac{1}{2}}\|_{H^{\frac{1}{2}+\epsilon}(f)} \\
&\le C\epsilon^{-2} p^{-(r-(\frac{1}{2}+\epsilon))} \sum_f \|u\|_{H^r(f)} \\
&\le C(\ln p)^2 p^{-(r-\frac{1}{2})} \sum_f \|u\|_{H^r(f)} \\
&\le C(\ln p)^2 p^{-(r-\frac{1}{2})} \|u\|_{H^r(\partial\Omega)}
\end{aligned} \tag{4.189}$$

Notice that one of the $O(\epsilon^{-\frac{1}{2}})$ contributions comes from the use of the Poincare's inequality. This could have been avoided if the $H^{\frac{1}{2}}$-norm rather then $H^{\frac{1}{2}}$-seminorm were used in the projection over faces, comp. [12, p.365].

We discuss next the localization argument for spaces $\boldsymbol{H}^s(\mathrm{curl}, \Omega)$. First of all, Proposition 4 generalizes to the multidimensional case. We have,

$$\|v\|_{H^{-t}(\partial\Omega)} \le C \sum_f \|v\|_{H^{-t}(f)}, \quad \forall v \in H^{-t}(\partial\Omega) \tag{4.190}$$

where $0 \le t < \frac{1}{2}$, and constant $C = O(\epsilon^{-1})$ for $t = \frac{1}{2} - \epsilon$. For $\boldsymbol{E}_t \in \boldsymbol{H}^{-t}(\mathrm{curl}, \partial\Omega)$ (see [11] for precise definitions) this immediately implies that,

$$\|\boldsymbol{E}_t\|_{\boldsymbol{H}^{-t}(\partial\Omega)} \le C \sum_f \|\boldsymbol{E}_t\|_{\boldsymbol{H}^{-t}(f)} \tag{4.191}$$

and,

$$\|\,\mathrm{curl}\, \boldsymbol{E}_t\|_{H^{-t}(\partial\Omega)} \le C \sum_f \|\,\mathrm{curl}\, \boldsymbol{E}_t\|_{H^{-t}(f)} \tag{4.192}$$

It remains only to argue that the restriction of the curl coincides with the curl of the restriction, i.e.,

$$\mathrm{curl}(\boldsymbol{E}_t|_f) = (\mathrm{curl}\, \boldsymbol{E}_t)|_f \tag{4.193}$$

But this follows immediately from the definition of the distributional derivatives. The last identity remains true for the limiting case $r = \frac{1}{2}$, comp. [29, p.104].

Using the same arguments as for the H^s-norms, we obtain the following estimates.

$$\begin{aligned}
\|\boldsymbol{E} - \Pi^{curl}_{s-1} \boldsymbol{E}\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},\partial\Omega)} &\le C \sum_f \|\boldsymbol{E} - \Pi^{curl}_{s-1} \boldsymbol{E}\|_{\boldsymbol{H}^{s-1}(\mathrm{curl},f)} \\
&\le C p^{-(r-s)} \sum_f \|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathrm{curl},f)} \\
&\le C p^{-(r-s)} \|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathrm{curl},\partial\Omega)} \\
\|\boldsymbol{E} - \Pi^{curl}_{-\frac{1}{2}} \boldsymbol{E}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)} &\le C(\ln p)^2 p^{-(r-\frac{1}{2})} \|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathrm{curl},\partial\Omega)}
\end{aligned} \tag{4.194}$$

Finally, we have the same results for the last operator.

$$
\begin{aligned}
\|v-\Pi_{s-1}v\|_{H^{s-1}(\partial\Omega)} &\le C\sum_f \|v-\Pi_{s-1}v\|_{H^{s-1}(f)} \\
&\le Cp^{-(r-s)}\sum_f \|v\|_{H^{r-1}(f)} \\
&\le Cp^{-(r-s)}\|v\|_{H^{r-1}(\partial\Omega)} \\
\|v-\Pi_{-\frac{1}{2}}v\|_{H^{-\frac{1}{2}}(v\partial\Omega)} &\le C(\ln p)^2 p^{-(r-\frac{1}{2})}\|v\|_{H^{r-1}(\partial\Omega)}
\end{aligned}
\tag{4.195}
$$

5 Commuting Projections and Projection-Based Interpolation Operators in Three Space Dimensions

5.1 Definitions and commutativity

We shall consider the following diagram.

$$
\begin{array}{ccccccccccccc}
I\!R & \longrightarrow & H^r(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}^{r-1}(\mathbf{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}^{r-1}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & H^{r-1}(\Omega) & \longrightarrow & \{0\} \\
\big\downarrow & & P^{grad}\big\downarrow\Pi^{grad} & & P^{curl}\big\downarrow\Pi^{curl} & & P^{div}\big\downarrow\Pi^{div} & & P\big\downarrow & & \big\downarrow \\
I\!R & \longrightarrow & W_p & \xrightarrow{\nabla} & \boldsymbol{Q}_p & \xrightarrow{\nabla\times} & \boldsymbol{V}_p & \xrightarrow{\nabla\cdot} & Y_p & \longrightarrow & \{0\}
\end{array}
\tag{5.196}
$$

Here $r > \frac{3}{2}$, $\nabla\times$ denotes the vector-valued curl operator, and $\nabla\cdot$ is the scalar-valued divergence operator. By $\boldsymbol{H}^{r-1}(\mathbf{curl},\Omega)$ we understand the space of all vector-valued functions in $\boldsymbol{H}^{r-1}(\Omega)$ whose curl is in $H^{r-1}(\Omega)$. Ω stands for a 3D element, a hexahedron, prism or tetrahedron, and $Q_p, \boldsymbol{W}_p, \boldsymbol{V}_p, Y_p$ denote any of the exact polynomial sequences defined on the element Ω, discussed in Section 2. The common property of those sequences is that the corresponding trace spaces for $Q_p, \boldsymbol{W}_p, \boldsymbol{V}_p$ corresponding to any face f, define 2D exact polynomial sequences discussed in the previous section.

The projection operators P^{grad}, P^{curl} and P^{div} are defined as follows.

$$
\left\{
\begin{array}{l}
P^{grad}u =: u_p \in W_p \\
\|\nabla u_p - \nabla u\|_{L^2(\Omega)} \to \min \\
(u_p - u, 1)_{L^2(\Omega)} = 0
\end{array}
\right.
\tag{5.197}
$$

$$
\left\{
\begin{array}{l}
P^{curl}\boldsymbol{E} =: \boldsymbol{E}_p \in \boldsymbol{Q}_p \\
\|\nabla\times\boldsymbol{E}_p - \nabla\times\boldsymbol{E}\|_{L^2(\Omega)} \to \min \\
(\boldsymbol{E}_p - \boldsymbol{E}, \nabla\phi)_{L^2(\Omega)} = 0,\ \forall\phi\in W_p
\end{array}
\right.
\tag{5.198}
$$

$$\begin{cases} P^{div}\boldsymbol{v} =: \boldsymbol{v}_p \in \boldsymbol{V}_p \\ \|\nabla\cdot\boldsymbol{v}_p - \nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)} \to \min \\ (\boldsymbol{v}_p - \boldsymbol{v}, \nabla\times\boldsymbol{\phi})_{L^2(\Omega)} = 0,\ \forall\boldsymbol{\phi}\in\boldsymbol{Q}_p \end{cases} \tag{5.199}$$

and P denotes the orthogonal projection onto Y_p in the L^2-norm.

Exercise 23 Show that the projections defined above make the diagram (5.196) commute. ∎

The projection-based interpolation operators are defined as follows.

$$\begin{cases} \Pi^{grad}u =: u_p \in W_p \\ u_p = \Pi^{grad}_{\frac{1}{2}}u \text{ on } \partial\Omega \\ \|\nabla u_p - \nabla u\|_{L^2(\Omega)} \to \min \end{cases} \tag{5.200}$$

$$\begin{cases} \Pi^{curl}\boldsymbol{E} =: \boldsymbol{E}_p \in \boldsymbol{Q}_p \\ \boldsymbol{E}_{t,p} = \Pi^{curl}_{-\frac{1}{2}}\boldsymbol{E}_t \text{ on } \partial\Omega \\ \|\nabla\times\boldsymbol{E}_p - \nabla\times\boldsymbol{E}\|_{L^2(\Omega)} \to \min \\ (\boldsymbol{E}_p - \boldsymbol{E}, \nabla\phi)_{L^2(\Omega)} = 0,\ \forall\phi\in W_p\ :\ \phi = 0 \text{ on } \partial\Omega \end{cases} \tag{5.201}$$

and

$$\begin{cases} \Pi^{div}\boldsymbol{v} =: \boldsymbol{v}_p \in \boldsymbol{V}_p \\ v_{n,p} = \Pi_{-\frac{1}{2}}v_n \text{ on } \partial\Omega \\ \|\nabla\cdot v_p - \nabla\cdot v\|_{L^2(\Omega)} \to \min \\ (\boldsymbol{v}_p - \boldsymbol{v}, \nabla\times\boldsymbol{\phi})_{L^2(\Omega)} = 0,\ \forall\boldsymbol{\phi}\in\boldsymbol{Q}_p\ :\ \phi_t = 0 \text{ on } \partial\Omega \end{cases} \tag{5.202}$$

Here $\Pi^{grad}_{\frac{1}{2}}, \Pi^{curl}_{-\frac{1}{2}}, \Pi_{-\frac{1}{2}}$ are the 2D interpolation operators discussed in the previous section, $\boldsymbol{E}_t, \boldsymbol{E}_{t,p}$ denote the tangential component of $\boldsymbol{E}, \boldsymbol{E}_p$, and $v_n, v_{n,p}$ denote the normal component of $\boldsymbol{v}, \boldsymbol{v}_p$ on the boundary $\partial\Omega$ respectively. Notice again that all minimization problems are constrained-minimization problems - the boundary values of the interpolants are fixed. Similarly to 1D and 2D, the projection operators can be interpreted as local minimization problems with Neumann boundary conditions, while the interpolation operators employ local Dirichlet boundary conditions implemented by means of the 2D interpolation operators.

Definition of tangential and normal traces on the boundary. For $\boldsymbol{E}, \boldsymbol{v} \in \boldsymbol{H}^r, \frac{1}{2} < r < \frac{3}{2}$, the tangential and normal components are understood in the sense of the Trace Theorem. The definition for the range $-\frac{1}{2} < r < \frac{1}{2}$ is again more complicated. The starting point for defining the normal component is the Gauss Theorem,

$$\int_\Omega \nabla\cdot\boldsymbol{v}\,\phi = -\int_\Omega \boldsymbol{v}\cdot\nabla\phi + \int_{\partial\Omega} v_n\,\phi \tag{5.203}$$

Taking a test function $\phi \in H^{\frac{1}{2}-r}(\partial\Omega)$, we consider an extension Φ that is bounded in $H^{1-r}(\Omega)$-norm by the $H^{\frac{1}{2}-r}(\partial\Omega)$ norm of ϕ. An argument identical to the one used when defining the tangential component E_t in the previous section, leads to the estimate

$$\|v_n\|_{H^{r-\frac{1}{2}}(\partial\Omega)} \leq C\left(\|\mathrm{div}\boldsymbol{v}\|_{H^s(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{H}^r(\Omega)}\right) \tag{5.204}$$

where $s > -\frac{1}{2}$ and can be taken equal to r. The estimate can again be considered as an equivalent of the Trace Theorem. Constant C is of order $O(\epsilon^{-\frac{1}{2}})$ for $r = -\frac{1}{2}+\epsilon$ or $r = \frac{1}{2}-\epsilon$. For $r = -\frac{1}{2}, \frac{1}{2}$, the normal trace cannot be defined.

The definition of the tangential component is more technical. Referring to [13, 11] for details, we sketch the main idea only. Again, we consider first a sufficiently regular field $\boldsymbol{E} \in \boldsymbol{H}^r(\Omega)$ and a test function $\boldsymbol{\phi} \in \boldsymbol{H}^{\frac{1}{2}-r}(\partial\Omega)$, to invoke the integration by parts formula:

$$\int_{\partial\Omega} \boldsymbol{E}_t(\boldsymbol{n}\times\boldsymbol{\phi}) = \int_\Omega (\nabla\times\boldsymbol{E})\boldsymbol{\Phi} - \int_\Omega \boldsymbol{E}(\nabla\times\boldsymbol{\Phi}) \tag{5.205}$$

Here $\boldsymbol{\Phi} \in \boldsymbol{H}^{1-r}(\Omega)$ is an extension of $\boldsymbol{\phi}$ (i.e. $\boldsymbol{n}\times\boldsymbol{\Phi}|_{\partial\Omega} = \boldsymbol{\phi}$) such that,

$$\|\boldsymbol{\Phi}\|_{\boldsymbol{H}^{1-r}(\Omega)} \leq C\|\boldsymbol{\phi}\|_{\boldsymbol{H}^{\frac{1}{2}-r}(\partial\Omega)} \tag{5.206}$$

The following estimate follows,

$$\|\boldsymbol{E}_t\|_{\boldsymbol{H}^{r-\frac{1}{2}}(\partial\Omega)} \leq C\left(\|\nabla\times\boldsymbol{E}\|_{\boldsymbol{H}^s(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}^r(\Omega)}\right) \tag{5.207}$$

where $-\frac{1}{2} < s < \frac{1}{2}$. Next we employ a special test function $\boldsymbol{\phi} = \nabla_{\partial K}\phi, \phi \in H^{\frac{1}{2}-s}(\partial\Omega)$, and consider an extension $\Phi \in H^{1-s}(\Omega)$ of potential ϕ. Integrating the boundary term by parts, we get,

$$\int_{\partial\Omega} (\mathrm{curl}_{\partial\Omega}\,\boldsymbol{E}_t)\phi = \int_{\partial\Omega} \boldsymbol{E}_t\nabla_{\partial\Omega}\times\phi = \int_\Omega (\nabla\times\boldsymbol{E})\nabla\Phi \tag{5.208}$$

This yields an additional estimate,

$$\|\,\mathrm{curl}_{\partial\Omega}\,\boldsymbol{E}_t\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \leq C\|\nabla\times\boldsymbol{E}\|_{\boldsymbol{H}^s(\Omega)} \tag{5.209}$$

Setting $s = r$ and combining the two estimates, we get a “Trace Theorem” for the $\boldsymbol{H}(\mathbf{curl})$ space.

$$\|\boldsymbol{E}_t\|_{\boldsymbol{H}^{r-\frac{1}{2}}(\mathrm{curl},\partial\Omega)} \leq C\left(\|\nabla\times\boldsymbol{E}\|_{\boldsymbol{H}^r(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}^r(\Omega)}\right) \tag{5.210}$$

The blow up of the constants prohibits extending the definition to values $s, r = -\frac{1}{2}, \frac{1}{2}$ and the constants are of order $O(\epsilon^{\frac{1}{2}})$ for $s, r = -\frac{1}{2}+\epsilon$ or $s, r = \frac{1}{2}-\epsilon$. In what follows, we shall use the inequalities (5.210) and (5.204) for the case of $r = 0$ only.

THEOREM 8

The interpolation operators make the diagram (5.196) commute. ∎

Proof: The comuttativity of the first block follows from the fact that operator Π_s^{grad} preserves constants. In order to show the commutativity of the second block, we need to demonstrate that,

$$\Pi^{curl}(\nabla u) = \nabla(\Pi^{grad} u) \tag{5.211}$$

Let $\boldsymbol{E} = \nabla u$. By the commutativity of the 2D diagram, we have,

$$\boldsymbol{E}_{t,p} = \Pi^{curl}_{-\frac{1}{2}} \nabla_{\partial\Omega} u = \nabla_{\partial\Omega} u_p \tag{5.212}$$

where $u_p = \Pi^{grad}_{\frac{1}{2}} u$ and $\nabla_{\partial\Omega}$ denotes the tangential gradient on the boundary of the element. Consequently,

$$\int_\Omega \nabla \times \boldsymbol{E}_p = \int_{\partial\Omega} \boldsymbol{n} \times \nabla_{\partial\Omega} u_p = 0 \tag{5.213}$$

At the same time,

$$(\mathrm{curl}\boldsymbol{E}_p, \mathrm{curl}\boldsymbol{F})_{L^2(\Omega)} = 0 \tag{5.214}$$

for every $\boldsymbol{F} \in \boldsymbol{Q}_p, \boldsymbol{F}_t = 0$ on $\partial\Omega$. However, the image of such polynomials $\boldsymbol{F}$ under the curl operator, coincides exactly with polynomials in $\boldsymbol{V}_p$ with zero average and zero divergence, where the curl of $\boldsymbol{E}_p$ lives. Consequently, $\mathrm{curl}\boldsymbol{E}_p = 0$ and $\boldsymbol{E}_p = \nabla u_p$ for some $u_p \in W_p$. Substituting ∇u_p into (5.201)$_4$ we learn that $u_p = \Pi^{grad} u$.

To prove the next commutativity property, we need to show that,

$$\Pi^{div}(\nabla \times \boldsymbol{E}) = \nabla \times (\Pi^{curl}\boldsymbol{E}) \tag{5.215}$$

Let $\boldsymbol{v} = \nabla \times \boldsymbol{E}$. By the commutativity of the 2D diagram, we have,

$$v_{n,p} = \Pi_{-\frac{1}{2}}(\mathrm{curl}_{\partial\Omega}\, \boldsymbol{E}_t) = \mathrm{curl}_{\partial\Omega}\, \boldsymbol{E}_{t,p} \tag{5.216}$$

where $\boldsymbol{E}_{t,p} = \Pi^{curl}_{-\frac{1}{2}} \boldsymbol{E}_t$ and $\mathrm{curl}_{\partial\Omega}$ denotes the surface curl on the boundary of the element. Consequently,

$$\int_\Omega \nabla \cdot \boldsymbol{v}_p = -\int_{\partial\Omega} v_{n,p} = -\int_{\partial\Omega} \mathrm{curl}_{\partial\Omega}\, \boldsymbol{E}_{t,p} = 0 \tag{5.217}$$

At the same time,

$$(\mathrm{div}\, \boldsymbol{v}_p, \mathrm{div}\, \boldsymbol{w})_{L^2(\Omega)} = 0 \tag{5.218}$$

for every $\boldsymbol{w} \in \boldsymbol{V}_p, w_n = 0$ on $\partial\Omega$. However, the image of such polynomials $\boldsymbol{w}$ under the div operator, coincides exactly with polynomials in Y_p with zero average, where the div of $\boldsymbol{v}_p$ lives. Consequently, $\mathrm{div}\, \boldsymbol{v}_p = 0$ and $\boldsymbol{v}_p = \nabla \times \boldsymbol{E}_p$ for some $\boldsymbol{E}_p \in \boldsymbol{Q}_p$. Substituting $\nabla \times \boldsymbol{E}_p$ into (5.202)$_4$, we learn that $\boldsymbol{E}_p = \Pi^{curl}\boldsymbol{E}$.

To prove the last commutativity property, we need to show that,

$$P(\nabla \cdot \boldsymbol{v}) = \nabla \cdot (\Pi^{div}\boldsymbol{v}) \tag{5.219}$$

By the commutativity of the 2D diagram, we have,

$$\int_\Omega \operatorname{div}(\boldsymbol{v}_p - \boldsymbol{v}) = -\int_{\partial\Omega} \boldsymbol{n} \cdot (\Pi^{div}\boldsymbol{v} - \boldsymbol{v}) = -\int_{\partial\Omega} (\Pi_{-\frac{1}{2}} v_n - v_n) = 0 \tag{5.220}$$

At the same time, the image of functions from $\boldsymbol{V}_p$ with zero normal traces coincides with the subspaces of functions from Y_p with zero average. Combining (5.220) with (5.202)$_4$ we obtain that,

$$\int_\Omega \operatorname{div}(\boldsymbol{v}_p - \boldsymbol{v})\, w_p = 0, \quad \forall w_p \in Y_p \tag{5.221}$$

Thus (5.219) holds. ∎

5.2 Polynomial preserving extension operators

We shall postulate the existence of commuting and polynomial preserving extension operators,

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^1(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}(\mathbf{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}(\operatorname{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\
 & & \Big\uparrow Ext & & \Big\uparrow Ext^{curl} & & \Big\uparrow Ext^{div} & & \\
\mathbb{R} & \longrightarrow & H^{\frac{1}{2}}(\partial\Omega) & \xrightarrow{\nabla_{\partial\Omega}} & \boldsymbol{H}^{-\frac{1}{2}}(\operatorname{curl},\partial\Omega) & \xrightarrow{\operatorname{curl}_{\partial\Omega}} & H^{-\frac{1}{2}}(\partial\Omega) & \xrightarrow{\int_{\partial\Omega}} & \mathbb{R}
\end{array} \tag{5.222}$$

that are right inverses of the trace operators for the energy spaces $H^1(\Omega), \boldsymbol{H}(\mathbf{curl},\Omega), \boldsymbol{H}(\operatorname{div},\Omega)$.

I am aware of four existing contributions addressing the existence of 3D polynomial preserving, extension operators for the H^1-space. Munoz-Sola [30] constructed such an operator for a tetrahedral element, and Bernardi, Dauge and Maday [5] provided such an operator for a cube. An elementary construction for a tetrahedron is shown in [36]. The discrete harmonic extensions studied by Pavarino and Widlund in [34], combined with the fundamental results on equivalence of continuous and discrete $H^{\frac{1}{2}}$-norms from [5] are the basis of a different construction for a cube in [14]. Construction of analogous extension operators for the $\boldsymbol{H}(\mathbf{curl},\Omega)$ and $\boldsymbol{H}(\operatorname{div},\Omega)$-spaces seems to be feasible but has not been done yet. We hope to report the construction of such operators for all three energy spaces and the tetrahedron in a forthcoming paper [36].

Thus, all reasoning presented in this chapter is contingent under results from [36] and the existence of analogous extensions for the hexahedral or prismatic elements.

5.3 Polynomial preserving, right-inverses of grad, curl, and div operators. Discrete Friedrichs inequalities

The following three operators, sometimes known as Poincare's maps, have been studied in [22].

$$
\begin{aligned}
&G \; : \boldsymbol{H}(\mathbf{curl},\Omega) \to H^1(\Omega) && (G\boldsymbol{E})(\boldsymbol{x}) = \boldsymbol{x} \cdot \int_0^1 \boldsymbol{E}(t\boldsymbol{x})\, dt \\
&K \; : \boldsymbol{H}(\mathrm{div},\Omega) \to \boldsymbol{H}(\mathbf{curl},\Omega) && (K\boldsymbol{v})(\boldsymbol{x}) = -\boldsymbol{x} \times \int_0^1 t\boldsymbol{v}(t\boldsymbol{x})\, dt \\
&D \; : L^2(\Omega) \to \boldsymbol{H}(\mathrm{div},\Omega) && (Dw)(\boldsymbol{x}) = \boldsymbol{x} \int_0^1 t^2 w(t\boldsymbol{x})\, dt
\end{aligned}
\tag{5.223}
$$

Exercise 24 Prove the following statements.

- The operators G, K, D satisfy the following identities,

$$
\begin{aligned}
&w = \nabla \cdot (Dw), && \forall w \in L^2(\Omega) \\
&\boldsymbol{v} = \nabla \times (K\boldsymbol{v}) + D(\nabla \cdot \boldsymbol{v}), && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \\
&\boldsymbol{E} = \nabla(G\boldsymbol{E}) + K(\nabla \times \boldsymbol{E}), && \forall \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl},\Omega)
\end{aligned}
\tag{5.224}
$$

- This implies that operators G, K, D, restricted to the range of operators grad, curl and div, respectively, are their right-inverses, i.e.,

$$
\begin{aligned}
&\nabla \cdot (Dw) = w, && \forall w \in L^2(\Omega) \\
&\nabla \times (K\boldsymbol{v}) = \boldsymbol{v}, && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \; : \; \nabla \cdot \boldsymbol{v} = 0 \\
&\nabla(G\boldsymbol{E}) = \boldsymbol{E}, && \forall \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl},\Omega) \; : \; \nabla \times \boldsymbol{E} = \boldsymbol{0}
\end{aligned}
\tag{5.225}
$$

- Prove that the operators are continuous (you may assume that Ω is the master tetrahedron, hexahedron, or prism).

- Prove that, for all discussed[7] elements: the Nédélec tetrahedra of the first and second types, the Nédélec hexahedron of constant polynomial order, operators G, K, D map the corresponding "face element" spaces into the "edge element" spaces.

∎

Lemma 4
(Discrete Friedrichs Inequalities for $\boldsymbol{H}(\mathbf{curl},\Omega)$ space)

[7] The result is true for the prismatic elements as well.

Let Ω be the master tetrahedron or hexahedron of uniform order p. There exists a constant $C > 0$ such that,

$$\|\boldsymbol{E}\|_{L^2(\Omega)} \leq C\|\nabla \times \boldsymbol{E}\|_{L^2(\Omega)} \, , \tag{5.226}$$

for every discrete divergence free polynomial $\boldsymbol{E} \in \boldsymbol{Q}_p$ belonging to one of the two families,
Case 1:

$$(\boldsymbol{E}, \nabla\phi)_{L^2(\Omega)} = 0, \quad \forall \phi \in W_p \tag{5.227}$$

Case 2: $\boldsymbol{E}_t = \boldsymbol{0}$ *on* $\partial\Omega$, *and*

$$(\boldsymbol{E}, \nabla\phi)_{L^2(\Omega)} = 0, \quad \forall \phi \in W_p \; : \; \phi = 0 \text{ on } \partial\Omega \tag{5.228}$$

▮

Proof: **Case 1** follows immediately from the continuity of the right-inverse K,

$$\begin{aligned}
\|\boldsymbol{E}\|_{L^2(\Omega)} &= \inf_{\phi\in W_p} \|\boldsymbol{E} - \nabla\phi\|_{L^2(\Omega)} \\
&\leq \|\boldsymbol{E} - (\boldsymbol{E} - K(\nabla \times \boldsymbol{E}))\|_{L^2(\Omega)} \\
&\leq C\|K(\nabla \times \boldsymbol{E})\|_{L^2(\Omega)} \\
&\leq C\|\nabla \times \boldsymbol{E}\|_{L^2(\Omega)}
\end{aligned} \tag{5.229}$$

Case 2: In order to account for the homogeneous boundary conditions, operator K has to be modified. Let $E \in \boldsymbol{Q}_p$ be a divergence free polynomial with zero trace. Consider $\boldsymbol{E} - K(\nabla \times \boldsymbol{E})$ where K is the right-inverse of the curl operator defined above. There exists then a polynomial $\psi \in W_p$ such that, $\boldsymbol{E} - K(\nabla \times \boldsymbol{E}) = \nabla\psi$. Then,

$$\boldsymbol{n} \times (\nabla\psi) = -\boldsymbol{n} \times (K(\nabla \times \boldsymbol{E})) \tag{5.230}$$

as $\boldsymbol{E}$ has a zero trace. Let $\Psi = Ext(tr\psi)$ where Ext is the polynomial preserving extension operator for the H^1-space. We have,

$$\|\nabla\Psi\|_{L^2(\Omega)} \leq C\|\psi\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|\boldsymbol{n} \times (\nabla\psi)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\|K(\nabla \times \boldsymbol{E})\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\|\nabla \times \boldsymbol{E}\|_{L^2(\Omega)} \tag{5.231}$$

Finally,

$$\begin{aligned}
\|\boldsymbol{E}\|_{L^2(\Omega)} &= \inf_{\phi\in W_p, \phi=0 \text{ on } \partial\Omega} \|\boldsymbol{E} - \nabla\phi\|_{L^2(\Omega)} \\
&\leq \|\boldsymbol{E} - (\boldsymbol{E} - K(\nabla \times \boldsymbol{E}) - \nabla\Psi)\|_{L^2(\Omega)} \\
&\leq C\|K(\nabla \times \boldsymbol{E}) + \nabla\Psi\|_{L^2(\Omega)} \\
&\leq C\|\nabla \times \boldsymbol{E}\|_{L^2(\Omega)}
\end{aligned} \tag{5.232}$$

The crucial fact is that the correction $\nabla\Psi$ is controlled only by the trace of the $K(\nabla \times \boldsymbol{E})$ and, consequently, by the L^2-norm of $\nabla \times \boldsymbol{E}$ only. ▮

A generalization to elements of variable order is non-trivial. The following reasoning has been put forth in [19] for the case of the tetrahedral element of variable order with an additional technical assumption that the polynomial order for the element edges is set to the minimum of the order for neighboring faces. The idea is based on the observation that the operator K preserves some of the local properties of the polynomial order of approximation. Let $\boldsymbol{v} \in \boldsymbol{V}_p, \nabla \cdot \boldsymbol{v} = 0$. It is easy to verify that the tangential components of $K\boldsymbol{v}$ along the coordinate axes are zero and that the order of $K\boldsymbol{v}$ for three faces neighboring the origin matches the order of approximation in space $\boldsymbol{V}_p$ (increased by one). The order of the sloped face is implied by the order of the (interior of the) tetrahedron. By using the Piola transformation (2.60), we extend the construction of operator to other vertices of the tetrahedron arriving at four maps $K_i, i = 0, \ldots, 3$, each corresponding to one of the vertices.

The second observation is the possibility of a stable decomposition[8]

$$\boldsymbol{v} = \boldsymbol{v}_0 = \sum_{i=1}^{4} \boldsymbol{v}_i, \text{ where } \quad \boldsymbol{v}_i \cdot \boldsymbol{n} = 0 \text{ on face } f_j, j = 1, \ldots, i, \quad i = 1, \ldots, 4 \tag{5.233}$$

and $\boldsymbol{v}_0$ has a zero normal trace. Additionally,

$$\|\boldsymbol{v}_i\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}, \quad i = 0, \ldots, 4 \tag{5.234}$$

Such a decomposition follows from the construction of polynomial-preserving extension operators for the tetrahedron, and the possibility of extending boundary values from one, two, three, and four faces (one-face, two-face, etc. extension operators). The logic of the decomposition is as follows. We pick a face f_1. Then,

$$\|v_n\|_{H^{-\frac{1}{2}}(f_1)} \leq \|v_n\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \tag{5.235}$$

Let $\boldsymbol{v}_1$ be then a stable extension of the restriction of v_n to face f_1. We subtract $\boldsymbol{v}_1$ from $\boldsymbol{v}$ and apply the same procedure to the union of two faces $f_1 \cup f_2$. Continuing in this manner we end up with the decomposition above.

Enumerating the faces in the order of increasing polynomial order for the normal component on the face, we construct the final right-inverse in the form,

$$K\boldsymbol{v} = \sum_{i=1}^{4} K_i \boldsymbol{v}_i + K_0 \boldsymbol{v}_0 \tag{5.236}$$

where $\boldsymbol{K}_0$ is the operator constructed in the proof of Lemma 4 for the case with homogeneous boundary conditions. Notice that each component $\boldsymbol{v}_i$ shares the order of the face and polynomial $K_i\boldsymbol{v}_i$ has zero trace on faces of lower order.

The idea does not extend to the hedrahedral element.

Lemma 5

(Discrete Friedrichs Inequality for $\boldsymbol{H}(\mathrm{div})$ space)

[8] Note that the decomposition into components with trace vanishing on all but one face, is impossible.

Let Ω be a tetrahedral or hexahedral element of an arbitrary variable order. There exists a constant $C > 0$ such that,

$$\|\boldsymbol{v}\|_{L^2(\Omega)} \leq C\|\nabla \circ \boldsymbol{v}\|_{L^2(\Omega)} \,, \tag{5.237}$$

for every discrete curl free polynomial $\boldsymbol{v}$ belonging to either of the two families:
Case 1: *$\boldsymbol{v} \in \boldsymbol{V}_p$ and,*

$$(\boldsymbol{v}, \nabla \times \boldsymbol{\phi})_{L^2(\Omega)} = 0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{Q}_p \,. \tag{5.238}$$

Case 2: *$\boldsymbol{v} \in \boldsymbol{V}_p$, $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, and,*

$$(\boldsymbol{v}, \nabla \times \boldsymbol{\phi})_{L^2(\Omega)} = 0, \quad \forall \boldsymbol{\phi} \in \boldsymbol{Q}_p \ : \ \boldsymbol{\phi}_t = 0 \text{ on } \partial\Omega \,. \tag{5.239}$$

Constant C is independent of polynomial order p. ∎

Proof: Let $w \in L^2(\Omega)$. We begin by decomposing w into a constant and a function with zero average,

$$w = c + w_0, \quad \int_\Omega w_0 = 0 \tag{5.240}$$

The normal component of function Dw_0 vanishes at the three faces adjacent to the origin but it may be non-zero on the remaining faces. Notice that the normal trace of Dw_0 has a zero average over the boundary and, therefore, it may be identified with a (surface) curl of an element from the trace space of $\boldsymbol{Q}_p$. Due to the commutativity of extension operators (5.222), extension $Ext^{div}TrDw_0$ has zero divergence. Consequently, map,

$$w_0 \to Dw_0 - Ext^{div}TrDw_0 \tag{5.241}$$

maps functions of zero average into functions with zero trace, and it is bounded. Let $\boldsymbol{v}_c \in \boldsymbol{V}_p$ be now any first order polynomial with divergence equal one. Map,

$$D^{mod}w = c\boldsymbol{v}_c + Dw_0 - Ext^{div}TrDw_0 \tag{5.242}$$

is a bounded, polynomial preserving right-inverse of the div operator. We conclude,

$$\begin{aligned} \|\boldsymbol{v}\|_{L^2(\Omega)} &= \inf_{\boldsymbol{\phi}} \|\boldsymbol{E} - \nabla \times \boldsymbol{\phi}\|_{L^2(\Omega)} \\ &\leq \|\boldsymbol{v} - (\boldsymbol{v} - D^{mod}(\nabla \cdot \boldsymbol{v}))\|_{L^2(\Omega)} \\ &\leq C\|D^{mod}(\nabla \cdot \boldsymbol{v})\|_{L^2(\Omega)} \\ &\leq C\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \end{aligned} \tag{5.243}$$

In case 1, the infimum is taken over the whole space $\boldsymbol{W}_p$, in case 2, the infimum is taken over the subspace of $\boldsymbol{W}_p$ of functions with zero trace. ∎

5.4 Projection and interpolation error estimates

We record yet the classical result.

Lemma 6
(Poincare's inequalities)

There exist $C > 0$ such that,

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \; ,$$

for every function $u \in H^1(\Omega)$ belonging to either of the two families:
Case 1: $(u, 1)_{L^2(\Omega)} = 0$,
Case 2: $u = 0$ *on* $\partial\Omega$. $\blacksquare$

In particular, both inequalities hold on the discrete level, for polynomials $u \in W_p$.

Equipped with the Poincare and discrete Friedrichs inequalities, as well as the conjectured extension operators, we can reproduce the arguments used in the previous sections, to conclude the error estimates for both the commuting projection and the projection-based interpolation operators.

Determining each of the projections $P^{grad}u, P^{curl}\boldsymbol{E}, P^{div}\boldsymbol{v}$ can be interpreted as the solution of a constrained minimization problem that leads to a mixed formulation with the Lagrange multiplier equal zero.

Exercise 25 Write out the mixed formulations corresponding to the definition of the commuting projections. $\blacksquare$

In each of the three cases, the first Brezzi's inf-sup condition is automatically implied by the discrete exact sequence property, with constant $\beta = 1$. The inf-sup in kernel condition is implied by the Poincare and discrete Friedrichs inequalities, case 1 (with no boundary conditions).

Equivalently, one can use the constructed right-inverses of operators grad, curl, div, to prove those inequalities,

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}(\mathrm{curl}, \Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}(\mathrm{div}, \Omega) & \xrightarrow{\nabla\circ} & L^2(\Omega) \\
P^{grad}\Big\downarrow P^{grad0} & & P^{curl}\Big\downarrow P^{curl0} & & P^{div}\Big\downarrow P^{div0} & & \Big\downarrow P \\
W_p & \xleftarrow{G} & \boldsymbol{Q}_p & \xleftarrow{K} & \boldsymbol{V}_p & \xleftarrow{D} & Y_p
\end{array} \tag{5.244}$$

Here $P^{grad0}, P^{curl0}, P^{div0}$ denote projections onto the subspaces of polynomials with zero grad, curl or div, i.e. on constants, gradients and curls of polynomials. With those inverses in hand, one can represent the projection operators in somehow less intuitive but more compact form [22],

$$\begin{aligned} P^{div}\boldsymbol{F} \quad &= P_0^{div}(\boldsymbol{F} - DP(\nabla \circ \boldsymbol{F})) + DP(\nabla \circ \boldsymbol{F}) \\ P^{curl}\boldsymbol{E} \quad &= P_0^{curl}(\boldsymbol{E} - KP_0^{div}(\nabla \times \boldsymbol{E})) + KP_0^{div}(\nabla \times \boldsymbol{E}) \\ P^{grad}\boldsymbol{F} \quad &= P_0^{grad}(u - GP_0^{curl}(\nabla u)) + GP_0^{curl}(\nabla u)\ . \end{aligned} \tag{5.245}$$

The representations imply the continuity of the commuting projections. As all of them are also preserving the polynomial spaces, this implies in turn their optimality - the projection errors can be bounded by the *best approximation errors* in norms $H^1, \boldsymbol{H}(\text{curl}), \boldsymbol{H}(\text{div})$, respectively.

THEOREM 9

There exist constants $C > 0$, independent of p such that,

$$\begin{aligned} \|u - P^{grad}u\|_{H^1(\Omega)} \quad &\leq C \inf_{u_p \in W_p} \|u - u_p\|_{H^1(\Omega)} \quad &&\leq Cp^{-(r-1)}\|u\|_{H^r(\Omega)}, \\ & &&\forall u \in H^r(\Omega),\ r > 1 \\ \|\boldsymbol{E} - P^{curl}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \quad &\leq C \inf_{\boldsymbol{E}_p \in \boldsymbol{Q}_p} \|\boldsymbol{E} - \boldsymbol{E}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \quad &&\leq Cp^{-(r-1)}\|\boldsymbol{E}\|_{\boldsymbol{H}^{r-1}(\mathbf{curl},\Omega)}, \\ & &&\forall \boldsymbol{E} \in \boldsymbol{H}^{r-1}(\mathbf{curl},\Omega),\ r > 1 \\ \|\boldsymbol{v} - P^{div}\boldsymbol{v}\|_{\boldsymbol{H}(\text{div},\Omega)} \quad &\leq C \inf_{\boldsymbol{v}_p \in \boldsymbol{V}_p} \|\boldsymbol{v} - \boldsymbol{v}_p\|_{\boldsymbol{H}(\text{div},\Omega)} \quad &&\leq Cp^{-(r-1)}\|\boldsymbol{v}\|_{\boldsymbol{H}^{r-1}(\text{div},\Omega)}, \\ & &&\forall \boldsymbol{v} \in \boldsymbol{H}^{r-1}(\text{div},\Omega),\ r > 1 \end{aligned} \tag{5.246}$$

▮

Proof: For the best approximation results, see [37]. ▮

The reasoning leading to the interpolation error estimates is identical for all three cases. We shall discuss the $\boldsymbol{H}(\mathbf{curl})$ case, with the remaining two being fully analogous. We begin by comparing the projection and interpolation errors.

$$\|\boldsymbol{E} - \Pi^{curl}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq \|\boldsymbol{E} - P^{curl}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|P^{curl}\boldsymbol{E} - \Pi^{curl}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \tag{5.247}$$

It follows from the definitions of the projection and interpolation operators that function $\boldsymbol{\psi} = P^{curl}\boldsymbol{E} - \Pi^{curl}\boldsymbol{E}$ is a discrete divergence-free, minimum energy extension of its boundary values, with the energy measured using the L^2-norm of the curl. The discrete Friedrichs inequality, case 2, implies that the $\boldsymbol{H}(\mathbf{curl})$-norm of $\boldsymbol{\psi}$ is bounded by the norm of an analogous discrete divergence-free, minimum energy extension with the energy measured using the full $\boldsymbol{H}(\mathbf{curl})$-norm. Indeed, let $\boldsymbol{\phi}$ be such an extension. We have,

$$\begin{aligned}
\|\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} &\le C(\|\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{\psi}\|_{L^2(\Omega)}) \\
&\le C(\|\boldsymbol{\psi}-\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{\psi}\|L^2_(\Omega)) \\
&\le C(\|\nabla\times(\boldsymbol{\psi}-\boldsymbol{\phi})\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{\phi}\|_{L^2(\Omega)}) \\
&\le C(\|\nabla\times\boldsymbol{\psi}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{\phi}\|_{L^2(\Omega)}) \\
&\le C(\|\nabla\times\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{L^2(\Omega)} + \|\nabla\times\boldsymbol{\phi}\|_{L^2(\Omega)}) \\
&\le C\|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}
\end{aligned} \tag{5.248}$$

We can invoke now the argument with polynomial preserving extension operators (which use the full norms...) to arrive at the final conclusion.

$$\begin{aligned}
\|\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} &\le C\|Ext^{curl}\|\,\|Tr(P^{curl}\boldsymbol{E}-\Pi^{curl}\boldsymbol{E})\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)} \\
&\le C\|Ext^{curl}\|\left(\|Tr(P^{curl}\boldsymbol{E}-\boldsymbol{E})\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)} + \|Tr(\boldsymbol{E}-\Pi^{curl}\boldsymbol{E})\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)}\right) \\
&\le C\|Ext^{curl}\|\left(C_{tr}\|(P^{curl}\boldsymbol{E}-\boldsymbol{E})\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} + \|\boldsymbol{E}_t-\Pi^{curl}_{-\frac{1}{2}}\boldsymbol{E}_t)\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)}\right)
\end{aligned} \tag{5.249}$$

where C_{tr} is the trace constant corresponding to estimate (5.210).

Exercise 26 Reproduce the reasoning above for the H^1 and $\boldsymbol{H}(\mathrm{div})$-spaces. ∎

Combining the reasoning above with Theorem 9 and estimates (4.189), (4.194), (4.195), we get our final result.

THEOREM 10

There exist constants $C > 0$, independent of p such that,

$$\begin{aligned}
\|u-\Pi^{grad}u\|_{H^1(\Omega)} &\le C\left(\inf_{u_p\in W_p}\|u-u_p\|_{H^1(\Omega)} + \|u-\Pi^{grad}_{\frac{1}{2}}u)\|_{\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)}\right) \\
&\le C(\ln p)^2 p^{-(r-1)}\|u\|_{H^r(\Omega)}, \quad \forall u\in H^r(\Omega),\ r>\frac{3}{2} \\
\|\boldsymbol{E}-\Pi^{curl}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} &\le C\left(\inf_{\boldsymbol{E}_p\in\boldsymbol{Q}_p}\|\boldsymbol{E}-\boldsymbol{E}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\|\boldsymbol{E}_t-\Pi^{curl}_{-\frac{1}{2}}\boldsymbol{E}_t)\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)}\right) \\
&\le C(\ln p)^2 p^{-r}\|\boldsymbol{E}\|_{\boldsymbol{H}^r(\mathbf{curl},\Omega)}, \quad \forall \boldsymbol{E}\in\boldsymbol{H}^r(\mathbf{curl},\Omega),\ r>\frac{1}{2} \\
\|\boldsymbol{v}-\Pi^{div}\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} &\le C\left(\inf_{\boldsymbol{v}_p\in\boldsymbol{V}_p}\|\boldsymbol{v}-\boldsymbol{v}_p\|_{\boldsymbol{H}(\mathrm{div},\Omega)} + \|v_n-\Pi_{-\frac{1}{2}}v_n)\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right) \\
&\le C(\ln p)^2 p^{-r}\|\boldsymbol{v}\|_{\boldsymbol{H}^r(\mathrm{div},\Omega)}, \quad \forall \boldsymbol{v}\in\boldsymbol{H}^r(\mathrm{div},\Omega),\ r>0
\end{aligned} \tag{5.250}$$

$\blacksquare$

Remark 3 Since all interpolation operators are polynomial-preserving, the classical Bramble-Hilbert argument allows to generalize the p estimates in Theorem 10 to corresponding hp-estimates. $\blacksquare$

6 Application to Maxwell equations. Open problems

6.1 Time-harmonic Maxwell equations

We shall consider the time-harmonic Maxwell equations in a bounded domain $\Omega \subset \mathbb{R}^n,\ n = 2, 3$.

- Faraday's law,

$$\frac{1}{\mu}\nabla \times \boldsymbol{E} = -i\omega \boldsymbol{H} \ , \tag{6.251}$$

- Ampere's law,

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{imp} + \sigma \boldsymbol{E} + \epsilon i\omega \boldsymbol{E} \ . \tag{6.252}$$

Here μ, σ, ϵ denote the material data: permeability, conductivity and permittivity, assumed to be piecewise constant, ω is the angular frequency, and $\boldsymbol{J}^{imp}$ stands for the impressed current. We can derive two alternative variational formulations by choosing one of the equations to be satisfied in a weak, distributional sense, and the other one pointwise. The choice is dictated usually by the nature of source terms and/or boundary conditions. Choosing e.g. the Ampere's law to be satisfied in the weak sense, we multiply (6.252) with a test function $\boldsymbol{F}$, integrate over the domain and integrate by parts to obtain,

$$\int_\Omega \boldsymbol{H}\ \nabla \times \boldsymbol{F} + \int_{\partial\Omega} \boldsymbol{n} \times \boldsymbol{H}\ \boldsymbol{F} - \int_\Omega (\sigma \boldsymbol{E} + \epsilon i\omega \boldsymbol{E})\boldsymbol{F} = \int_\Omega \boldsymbol{J}^{imp}\boldsymbol{F} \tag{6.253}$$

Notice that equations (6.251) and (6.253) imply implicitly the satisfaction of the Gauss law for magnetism (in the strong sense) and the continuity equation (in the weak sense). Eliminating $\boldsymbol{H}$ using (6.251) and employing appropriate boundary conditions, we get the classical variational formulation.

$$\left\{ \begin{array}{l} \boldsymbol{E} \in H(\mathrm{curl}, \Omega),\ \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{n} \times \boldsymbol{E}_D \text{ on } \Gamma_D \\ \displaystyle\int_\Omega \left\{ \frac{1}{\mu}(\nabla \times \boldsymbol{E})(\nabla \times \boldsymbol{F}) - (\omega^2\epsilon - i\omega\sigma)\boldsymbol{E}\boldsymbol{F} \right\}\, d\boldsymbol{x} + i\omega \int_{\Gamma_C} \gamma \boldsymbol{E}_t \boldsymbol{F}\, dS \\ \displaystyle = -i\omega \int_\Omega \boldsymbol{J}^{imp}\boldsymbol{F}\, d\boldsymbol{x} + i\omega \int_{\Gamma_N \cup \Gamma_C} \boldsymbol{J}_S^{imp}\boldsymbol{F}\, dS \quad \text{for every } \boldsymbol{F} \in H(\mathrm{curl}, \Omega),\ \boldsymbol{n} \times \boldsymbol{F} = 0 \text{ on } \Gamma_D \ . \end{array} \right. \tag{6.254}$$

Here Γ_D, Γ_N and Γ_C stand for the parts of the boundary where the Dirichlet (perfect conductor), Neumann (prescribed magnetic current) and Cauchy (prescribed impedance) boundary conditions have been set up, $\boldsymbol{E}_D$ stands for the Dirichlet data, γ is the impedance constant, and $\boldsymbol{J}_S^{imp}$ is a surface current prescribed on both Neumann and Cauchy parts of the boundary.

Weak form of the continuity equation. Employing a special test function, $\boldsymbol{F} = \nabla q, q \in H^1(\Omega), q = 0$ on Γ_D, we learn that the solution to the variational problem satisfies automatically the weak form of the continuity equation,

$$\begin{aligned} &\int_\Omega -(\omega^2\epsilon - i\omega\sigma)\boldsymbol{E}\nabla q\, d\boldsymbol{x} + i\omega \int_{\Gamma_C} \gamma \boldsymbol{E}_t \nabla q\, dS \\ &\quad = -i\omega \int_\Omega \boldsymbol{J}^{imp} \nabla q\, d\boldsymbol{x} + i\omega \int_{\Gamma_N \cup \Gamma_C} \boldsymbol{J}_S^{imp} \nabla q\, dS \qquad \text{for every } q \in H^1(\Omega),\ q = 0 \text{ on } \Gamma_D\,. \end{aligned} \tag{6.255}$$

Upon integrating by parts, we learn that solution $\boldsymbol{E}$ satisfies the continuity equation,

$$\operatorname{div}\big((\omega^2\epsilon - i\omega\sigma)\boldsymbol{E}\big) = i\omega\, \operatorname{div}\boldsymbol{J}^{imp} \qquad (= \omega^2\rho)\,,$$

plus additional boundary conditions on Γ_N, Γ_C, and interface conditions across material interfaces.

Maxwell eigenvalue problem. Related to the time-harmonic problem (6.254) is the eigenvalue problem,

$$\left\{ \begin{array}{l} \boldsymbol{E} \in H(\text{curl}, \Omega),\ \boldsymbol{n} \times \boldsymbol{E} = 0 \text{ on } \Gamma_D, \quad \lambda \in \mathbb{R} \\ \displaystyle\int_\Omega \frac{1}{\mu} (\nabla \times \boldsymbol{E})(\nabla \times \boldsymbol{F})\, d\boldsymbol{x} = \lambda \int_\Omega \epsilon \boldsymbol{E}\boldsymbol{F}\, d\boldsymbol{x} \quad \text{for every } \boldsymbol{F} \in H(\text{curl}, \Omega),\ \boldsymbol{n} \times \boldsymbol{F} = 0 \text{ on } \Gamma_D\,. \end{array} \right. \tag{6.256}$$

The curl-curl operator is self-adjoint, its spectrum consists of $\lambda = 0$ with an infinite-dimensional eigenspace consisting of all gradients $\nabla p, p \in H^1(\Omega), p = 0$ on Γ_D, and a sequence of positive eigenvalues $\lambda_1 < \lambda_2 < \ldots \lambda_n \to \infty$ with corresponding eigenspaces of finite dimension. Only the eigenvectors corresponding to positive eigenvalues are physical. Repeating the reasoning with the substitution $\boldsymbol{F} = \nabla q$, we learn that they satisfy automatically the continuity equation.

Stabilized variational formulation. The standard variational formulation (6.254) is *not* uniformly stable with respect to frequency ω. As $\omega \to 0$, we loose the control over gradients. This corresponds to the fact that, in the limiting case $\omega = 0$, the problem is ill-posed as the gradient component remains undetermined. A remedy to this problem is to enforce the continuity equation explicitly at the expense of introducing a Lagrange multiplier p. The so called *stabilized variational formulation* looks as follows.

$$
\left\{
\begin{array}{l}
\boldsymbol{E} \in H(\mathrm{curl},\Omega), p \in H^1(\Omega), \quad \boldsymbol{n}\times\boldsymbol{E} = \boldsymbol{n}\times\boldsymbol{E}_0, p = 0 \text{ on } \Gamma_D, \\
\displaystyle\int_\Omega \frac{1}{\mu}(\nabla\times\boldsymbol{E})(\nabla\times\boldsymbol{F})d\boldsymbol{x} - \int_\Omega (\omega^2\epsilon - i\omega\sigma)\boldsymbol{E}\cdot\boldsymbol{F}d\boldsymbol{x} + i\omega\int_{\Gamma_C}\gamma\boldsymbol{E}_t\boldsymbol{F}\,dS \\
\displaystyle - \int_\Omega (\omega^2\epsilon - i\omega\sigma)\nabla p\cdot\boldsymbol{F}d\boldsymbol{x} = -i\omega\int_\Omega \boldsymbol{J}^{imp}\cdot\boldsymbol{F}d\boldsymbol{x} + i\omega\int_{\Gamma_N\cup\Gamma_C}\boldsymbol{J}_S^{imp}\cdot\boldsymbol{F}dS \\
\hfill \forall\boldsymbol{F}\in H(\mathrm{curl},\Omega),\ \boldsymbol{n}\times\boldsymbol{F} = 0 \text{ on } \Gamma_D, \\
\displaystyle - \int_\Omega (\omega^2\epsilon - i\omega\sigma)\boldsymbol{E}\cdot\nabla q\,d\boldsymbol{x} + i\omega\int_{\Gamma_C}\gamma\boldsymbol{E}_t\nabla q\,dS = \\
\hfill \displaystyle -i\omega\int_\Omega \boldsymbol{J}^{imp}\cdot\nabla q\,d\boldsymbol{x} + i\omega\int_{\Gamma_N}\boldsymbol{J}_S^{imp}\cdot\nabla q dS \\
\hfill \forall q\in H^1(\Omega),\ q = 0 \text{ on } \Gamma_D\,.
\end{array}
\right.
\tag{6.257}
$$

By repeating the reasoning with the substitution $\boldsymbol{F} = \nabla q$ in the first equation, we learn that the Lagrange multiplier p satisfies the weak form of a Laplace-like equation with homogeneous boundary conditions and, therefore, it *identically vanishes*. For that reason, it is frequently called the *hidden variable*. The stabilized formulation has improved stability properties for small ω. In the case of $\sigma = 0$ and right hand side of (6.255) vanishing, we can rescale the Lagrange multiplier, $p = \omega^2 p, q = \omega^2 q$, to obtain a symmetric mixed variational formulation with stability constant converging to one as $\omega \to 0$. In the general case we cannot avoid a degeneration as $\omega \to 0$ but we can still rescale the Lagrange multiplier with ω ($p = \omega p, q = \omega q$), to improve the stability of the formulation for small ω. The stabilized formulation is possible because gradients of the scalar-valued potentials from $H^1(\Omega)$ form precisely the null space of the curl-curl operator.

The point about the stabilized (mixed) formulation is that, whether we use it or not in the actual computations (the improved stability is one good reason to do it...), the original variational problem is *equivalent* to the mixed problem. This suggests that we cannot escape from the theory of mixed formulations when analyzing the problem.

6.2 So why does the projection-based interpolation matter ?

The classical result of the numerical analysis for linear problems states that *discrete stability and approximability imply convergence*. For Finite Element (FE) approximations of mixed problems this translates into the control of the two inf-sup constants and best approximation error estimates. As for the mixed formulations of commuting projections, the exact sequence property implies the automatic satisfaction of the first Brezzi's inf-sup condition, with constant $\beta = 1$. The exact sequence is now understood at the level of the whole FE mesh. The satisfaction of the inf-sup in kernel condition is implied by the convergence of Maxwell eigenvalues, see [21] for the analysis of the lossless case $\sigma = 0$, and [9, 6, 7] for the related work. In this context, the projection-based interpolation enters the picture in two places. The best approximation error (over the whole mesh) is estimated with the interpolation error for the exact solution. The minimal regularity assumptions allow for estimating the error for solutions of "real" problems exhibiting multiple singularities due to the presence of reentrant corners and edges, and material interfaces.

The second use of the projection-based interpolation error has been recorded in the only existing proof on the hp-convergence of Maxwell eigenvalues for 2D Nédélec quads of the second type and 1-irregular meshes with hanging nodes, see [8]. Contrary to the 3D case, the 2D interpolant $\Pi_0^{curl}\boldsymbol{E}$ requires only an increased regularity in the field itself, $\boldsymbol{E} \in \boldsymbol{H}^r,\ r > 0$, but with $\mathrm{curl}\boldsymbol{E} \in L^2(\Omega)$ only. This leads to the possibility of estimating the error in L^2-norm,

$$\|\boldsymbol{E} - \Pi_0^{curl}\boldsymbol{E}\| \le C(\frac{h}{p})^r(\|\boldsymbol{E}\|_{\boldsymbol{H}^r(\Omega)} + \|\mathrm{curl}\boldsymbol{E}\|_{L^2(\Omega)}) \tag{6.258}$$

The estimate does not follow from the classical duality argument and exceeds the scope of these notes. Its use has been essential in proving the discrete compactness result in [8] which leads to the convergence result for the Maxwell eigenvalue problem and, in the end, the stability result for the 2D time-harmonic Maxwell equations.

Finally, the projection-based interpolation has been the driving idea behind the fully automatic hp-adaptivity producing a sequence of hp meshes that deliver exponential convergence, see [17] and the literature there. The concept of the projection-based interpolation extends naturally to element patches and spaces of piecewise polynomials.

6.3 Open problems

We finish by summarizing the major open problems related to the theory of the projection-based interpolation and the grad-curl-div sequence for elements of higher order.

Extension operators. With the contribution [36] in place, the task of constructing the commuting, polynomial preserving extension operators for the hexahedral and prismatic elements, remains to be the main challenge. The second open issue is the construction of the right-inverse of the curl operator and hexahedron of an arbitrary order. The construction for the tetrahedral element should be freed from the technical assumption on the edge polynomial order.

Pyramids. A successful three-dimensional FE code for Maxwell equations must include all four kind of geometrical shapes: teds, hexas, prisms, and *pyramids*. The theory of the exact sequence and higher order elements for the pyramid element remains to be one of the most urgent research issues (for the lowest order edge pyramid, see the work of Gradinaru and Hiptmair [24]). The usual reasoning is to view the pyramid as a special case of the parametric element based on a singular map.

Discrete compactness, L^2-estimates, non-local interpolation. It is not clear whether the techniques used in [8], can be generalized to 3D. The fact that, in the discrete compactness argument, $\nabla \times \boldsymbol{E}$ lives only on $L^2(\Omega)$, eliminates the use of the projection-based interpolant in the argument analogous to the one used in [8]. The minimum regularity assumptions for the projection-based interpolation are identical with those for the classical Lagrange, Nédélec and Raviart-Thomas interpolants. The use of non-local interpolation techniques like the one proposed by Schoeberl [35] seems to be essential.

References

[1] M. Ainsworth and L. Demkowicz. Explicit polynomial preserving trace liftings on a triangle. *London Mathematical Society*, submitted. see also *ICES Report* 03-47.

[2] D.N. Arnold, B. Boffi, and R.S. Falk. Quadrilateral $H(div)$ finite element. *I.A.N.-C.N.R*, 1283:1–23, 2002. (http://www.imati.cnr.it/ian/PUBBLICAZIONI/Pubblicazioni2002.html).

[3] I. Babuška, A. Craig, J. Mandel, and J. Pitkaränta. Efficient preconditioning for the p-version finite element method in two dimensions. *SIAM J. Numer. Anal*, 28(3):624–661, 1991.

[4] I. Babuška and M. Suri. The optimal convergence rate of the p-version of the finite element method. *SIAM J. Numer. Anal.*, 24:750–776, 1987.

[5] C. Bernardi, M. Dauge, and Y. Maday. Polynomials in the Sobolev world. Technical Report R 03038, Laboratoire Jacques-Louis Lions, Université Pierre at Marie Curie, 2003. http://www.ann.jussieu.fr/.

[6] D. Boffi. Fortin operator and discrete compactness for edge elements. *Numer. Math.*, 87(2):229–246, 2000.

[7] D. Boffi. A note on the de Rham complex and a discrete compactness property. *Appl. Math. Lett.*, 14(1):33–38, 2001.

[8] D. Boffi, M. Dauge, M. Costabel, and L. Demkowicz. Discrete compactness for the hp version of rectangular edge finite elements. *SIAM J. on Numer. Anal.*, 2005. accepted, see also *ICES Report* 04-29.

[9] D. Boffi, P. Fernandes, L. Gastaldi, and I. Perugia. Computational models of electromagnetic resonators: analysis of edge element approximation. *SIAM J. Numer. Anal.*, 36(4):1264–1290, 1999.

[10] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers. *R.A.I.R.O.*, 8(R2):129–151, 1974.

[11] A. Buffa and P. Ciarlet. On traces for functional spaces related to Maxwell's equations. Part i: an integration by parts formula in Lipschitz polyhedra. *Mathematical Methods in the Applied Sciences*, 24:9–30, 2001.

[12] W. Cao and L. Demkowicz. Optimal error estimate for the projection based interpolation in three dimensions. *Comput. Math. Appl.*, 50:359–366, 2005.

[13] M. Cessenat. *Mathematical Methods in Electromagnetism*. World Scientific, Singapure, 1996.

[14] M. Costabel, M. Dauge, and L. Demkowicz. A study on polynomial extension operators for H^1, $\boldsymbol{H}(\mathbf{curl})$ and $\boldsymbol{H}(\mathrm{div})$ spaces on a cube. in preparation, 2006.

[15] L. Demkowicz. Edge finite elements of variable order for Maxwell's equations. In D. Hecht U. van Rienen, M. Günther, editor, *Scientific Computing in Electrical Engineering, Lecture Notes in Computational Science and Engineering* **18**, pages 15–34. Springer Verlag, Berlin, 2000. (Proceedings of the 3rd International Workshop, August 20-23, Warnemuende, Germany).

[16] L. Demkowicz. Projection based interpolation. In *Transactions on Structural Mechanics and Materials*. Cracow University of Technology Publications, Cracow, 2004. Monograph 302, A special issue in honor of 70th Birthday of Prof. Gwidon Szefer, see also *ICES Report* 04-03.

[17] L. Demkowicz. *Computing with hp Finite Elements. I.One- and Two-Dimensional Elliptic and Maxwell Problems*. CRC Press, Taylor and Francis, 2006.

[18] L. Demkowicz and I. Babuška. p interpolation error estimates for edge finite elements of variable order in two dimensions. *SIAM J. Numer. Anal.*, 41(4):1195–1208 (electronic), 2003.

[19] L. Demkowicz and A. Buffa. $H^1, H(\mathrm{curl})$ and $H(\mathrm{div})$-conforming projection-based interpolation in three dimensions. Quasi-optimal p-interpolation estimates. *Comput. Methods Appl. Mech. Engrg*, 194:267–296, 2005.

[20] L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz. De Rham diagram for hp finite element spaces. *Comput. Math. Appl.*, 39(7-8):29–38, 2000.

[21] L. Demkowicz and L. Vardapetyan. Modeling of electromagnetic absorption/scattering problems using hp-adaptive finite elements. *Comput. Methods Appl. Mech. Engrg.*, 152(1-2):103–124, 1998.

[22] J. Gopalakrishnan and L. Demkowicz. Quasioptimality of some spectral mixed methods. *Journal of Computational and Applied Mathematics*, 167(1), May 2004.

[23] J. Gopalakrishnan, L.E. García-Castillo, and L. Demkowicz. Nédélec spaces in affine ccordinates. *Comput. Math. Appl.*, 49:1285–1294, 2005.

[24] V. Gradinaru and R. Hiptmair. Whitney elements on pyramids. *ETNA*, 8:154–168, 1999. Also: Report 113, SFB 382, Universitt Tbingen, March 1999.

[25] P. Grisvard. *Singularities in boundary value problems*, volume 22 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1992.

[26] R. Hiptmair. Canonical construction of finite elements. *Mathematics of Computation*, 68, 1999. 1325-1346.

[27] R. Hiptmair. Higher order Whitney forms. Technical Report 156, Universität Tübingen, 2000.

[28] Y. Maday. Relévement de traces polynomiales et interpolations hilbertiennes entre espaces de polynomes. *C. R. Acad. Sci. Paris*, 309:463–468, 1989.

[29] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000.

[30] R. Munoz-Sola. Polynomial lifting on a tetrahedron and application to the hp-version of the finite element method. *SIAM J. Numer. Anal.*, 34:282–314, 1997.

[31] J.C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35:315–341, 1980.

[32] J.C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 50:57–81, 1986.

[33] J.T. Oden, L. Demkowicz, R. Rachowicz, and T.A. Westermann. Toward a universal hp adaptive finite element strategy. Part 2: a posteriori error estimation. *Comput. Methods Appl. Mech. Engrg.*, 77:113–180, 1989.

[34] L.F. Pavarino and O. B. Widlund. A polylogarithmic bound for iterative subtructuring method for spectral elements in three dimensions. *SIAM J. Numer. Anal.*, 33(4):1303–1335, 1996.

[35] J. Schoeberl. Commuting quasi-interpolation operators for mixed finite elements. Technical report, Texas A&M University, 2001. Preprint ISC-01-10-MATH.

[36] J. Schoeberl, J. Gopalakrishnan, and L. Demkowicz. Polynomial preserving H^1, $\boldsymbol{H}(\mathbf{curl})$ and $\boldsymbol{H}(\mathrm{div})$ extension operators on a tetrahedron. in preparation, 2006.

[37] Ch. Schwab. *p and hp-Finite Element Methods*. Clarendon Press, Oxford, 1998.

[38] J.P. Webb. Hierarchical vector based funtions of arbitrary order for triangular and tetrahedral finite elements. *IEEE Antennas Propagat. Mag.*, 47(8):1244–1253, 1999.

[39] H. Whitney. *Geometric Integration Theory*. Princeton University Press, 1957.