Stabilized methods for compressible flows

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Abstract

This article reviews 25 years of research on stabilized methods for compressible flow computations. An historical perspective is adopted to document the main advances from the initial developments to modern approaches.

Keywords: Stabilized methods, SUPG method, compressible flows.

1 Introduction

The development of stabilized methods began in the late 1970s. A number of papers appeared in conference proceedings and in books emanating from conferences. The first journal paper was that of Brooks and Hughes [5], which summarized early work on the subject. The application areas were advection–diffusion equations and the incompressible Navier–Stokes equations. The first stabilized method was SUPG, an acronym for streamline

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upwind/Petrov-Galerkin. At about the same time, the opportunity arose to extend the method to the compressible Euler equations. Airframe manufacturers and aerospace agencies, such as NASA, were making significant investments in computational fluid dynamics and this provided funding. This was occurring at the time shortly after one of us, T. Hughes, had joined Stanford University as a faculty member. Funding support from NASA Ames and NASA Langley was obtained and the challenge of developing a successful SUPG generalization began. Another one of us, T. Tezduyar, had come to Stanford from Caltech to write his thesis. In it Tezduyar developed the first finite element compressible flow formulation based on conservation variables (Tezduyar and Hughes [91, 92] and Hughes and Tezduyar [40]). During Tezduyar's postdoctoral stay at Stanford, Hughes and Tezduyar developed the first iterative solution strategy for the SUPG finite element computation of compressible flows utilizing the Element-by-Element (EBE) factorization of the coupled linear equation systems involved (Hughes, Winget, Levit and Tezduyar [41]).

A realization emerged from the first SUPG computation of compressible flows reported in [40, 91, 92] and that was that a more robust formulation would be needed to capture strong shocks. Work on this aspect of the problem was pursued in subsequent thesis work of Michel Mallet, in which entropy variables were also introduced. This provided a link with non-equilibrium thermodynamics (see the work of Hughes, Mallet and Franca [31]) and the first shock-capturing operators were developed (see Hughes, Mallet and Mizukami [35], Hughes and Mallet [34] and Tezduyar and Park [94]). Another contribution of this work was the refinement of the concept of the SUPG operator, facilitated by the use of entropy variables (see Hughes and Mallet [33]). During the time Mallet was performing his thesis research, Hughes and Mallet joined forces with Dassault Aviation to produce flow solvers that could be industrialized and put into production at Dassault. The main advocates of this collaboration on the Dassault side were Jacques Periaux and the world famous aeronautical engineer Pierre Perrier. This began a long and fruitful collaboration between the Stanford and Dassault teams.

Gerald Jay Le Beau, as part of his thesis work supervised by Tezduyar at University of Minnesota, revisited the original SUPG formulation of compressible flows introduced in [40, 91, 92] for conservation variables. Le Beau and Tezduyar [56] supplemented the formulation with a shock-capturing operator in conservation variables. For the shock-capturing parameter embedded in that operator, they used an expression in conservation variables, derived from the shock-capturing parameter embedded in the operator described in the above paragraph. They showed in [56] that, with the added shock-capturing operator, the original SUPG formulation of compressible flows in conservation variables is very comparable in accuracy to the SUPG formulation in entropy variables. Shortly after that, the 2D test computations for inviscid flows reported by Le Beau [55] showed that the SUPG formulation in conservation and entropy variables yielded indistinguishable results.

Mallet was followed at Stanford by Zdenek Johan and Frederic Chalot, all of whom became key players in the development of Dassaults production Navier–Stokes software capabilities. The three remain, as of this writing, at Dassault and are still active in the development of advanced capabilities and their application to the design of aircraft. Johan was a pioneer in the development of massively parallel flow solvers, as documented in Johan et al. [48, 49, 50], and Chalot demonstrated the superiority of the entropy variables formulation in chemically reacting flows. A summary of the research that has produced the Dassault flow solvers is presented in the Encyclopedia of Computational Mechanics article by Chalot [9]. Shakib pursued refinements of earlier stabilized method work in compressible flows in his thesis work and this was further developed in the commercial software Spectrum from Centric, and subsequently in other commercial software. Shakib's thesis research was published in [71, 72], and the work emphasized the second stabilized method to achieve popularity, namely, Galerkin/least-squares, or GLS (see Hughes, Franca and Hulbert [30]). Ken Jansen was the first to apply stabilized methods to turbulent compressible flows. In his thesis work, he developed an entropy-consistent formulation of the Norris–Reynolds RANS model (see Jansen, Johan and Hughes [46], Jansen and Hughes [45], and Jansen [44]). This work was followed by the thesis research of Guillermo Hauke who extended the ideas to the κ–ε turbulence model [23], and who generalized stabilized compressible flow methods to an arbitrary set of variables (see Hauke and Hughes [25, 26]). He also showed the utility of physical variables in transonic flows with shocks. This work once and for all dispelled myths from the finite difference literature that only conservation variables were appropriate for representing shock waves.

Over the years, significant progress was made by Tezduyar and his team at Minnesota, and later at Rice, on compressible flows. These include time-accurate local time stepping techniques [57], methods for flows with moving boundaries and interfaces [2, 81, 89, 90], methods for viscous flows [1], large-scale, parallel 3D computations [79, 80, 88], simulation of high-speed trains in relative motion [79], unified formulations for compressible and incompressible flows [60], shock capturing with multi-scale spatial discretization (two-level grid) [59], and new stabilization and shock-capturing parameters [85, 86, 95–97]. An overview of the stabilization and shock-capturing parameters, including these new ones, are given in Section 11.

Since the late 1990s the Boeing computational fluid dynamics team has been performing research on stabilized methods (see, e.g., Venkatakrishnan et al. [98]), and has more recently been developing production software.

Among the most recent work on compressible flows with stabilized methods is that of G. Scovazzi, a Stanford Ph.D. who completed his thesis work at UT Austin after Hughes moved there in 2002. Scovazzi's research was sponsored by Sandia Laboratories, and in 2004 he joined Sandia upon completion of his Ph.D. Scovazzi extended the SUPG formulation to very strong shocks in the context of Lagrangian hydrodynamics (with Mach numbers in the range 10^3–10^9). This work was significant in that it was the first successful formulation on unstructured triangular and tetrahedral Lagrangian meshes and the first of any kind for very strong shocks [69, 70].

2 The compressible Navier–Stokes problem

The compressible Navier–Stokes equations can be cast in system form as

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F} + \mathbf{G} = \mathbf{0} \;, \qquad \text{in } \Omega \subset \mathbb{R}^d \;,\; t > 0 \;, \tag{1}$$

$$\mathcal{U}(\mathbf{U}) = \mathbf{U}_g \;, \qquad \text{on } \partial\Omega^g \;,\; t > 0 \;, \tag{2}$$

$$\mathbf{F}\, \boldsymbol{n} = \mathbf{h} \;, \qquad \text{on } \partial\Omega^h \;,\; t > 0 \;, \tag{3}$$

$$\mathbf{U} = \mathbf{U}_0(\boldsymbol{x}) \;, \quad \text{in } \Omega \;,\; t = 0 \;. \tag{4}$$

Here, d indicates the number of space dimensions, ∂_t the Eulerian time derivative, $\partial\Omega^g$ the *Dirichlet* boundary. $\mathcal{U}(\cdot)$ is a boundary operator which, for the purpose of generality, may mask some of the entries of the vector

$$\mathbf{U} = \left\{ \begin{array}{c} \rho \\ \rho \boldsymbol{v} \\ \rho(e + \boldsymbol{v} \cdot \boldsymbol{v}/2) \end{array} \right\} \;. \tag{5}$$

$\mathbf{U}_g$ is the vector of Dirichlet boundary conditions, which, in the most general case, may be a function of the solution itself. Analogously, $\partial\Omega^h$ is the *Neumann* boundary, $\mathbf{h}$ is the $(n_d + 2)$-dimensional vector of Neumann conditions, and $\boldsymbol{n}$ is the unit outward normal vector on the boundary $\partial\Omega$. The $(n_d + 2) \times n_d$-matrix $\mathbf{F}$ is termed the flux matrix, and is defined as

$$\mathbf{F} = \mathbf{F}^{(c)} + \mathbf{F}^{(p)} + \mathbf{F}^{(d)} \;, \tag{6}$$

where

$$\mathbf{F}^{(c)} = \mathbf{U} \otimes \boldsymbol{v} \tag{7}$$

is the *convective* flux ($\mathbf{U} \otimes \boldsymbol{v} = \mathsf{U}_i v_j$),

$$\mathbf{F}^{(p)} = \begin{bmatrix} \mathbf{0}^T \\ p\boldsymbol{I} \\ \boldsymbol{v}^T p \end{bmatrix} \tag{8}$$

is the *pressure* flux, and

$$\mathsf{F}^{(d)} = - \begin{bmatrix} \mathbf{0}^T \\ 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v} \\ \boldsymbol{v}^T\,(2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v}) + \rho c_p \kappa (\nabla\theta)^T \end{bmatrix} \tag{9}$$

is the diffusive flux. Also,

$$\mathsf{G} = \begin{Bmatrix} 0 \\ \rho\boldsymbol{b} \\ \rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \end{Bmatrix} , \tag{10}$$

is the *source* term. In definitions (5)–(10), ρ is the density, $\boldsymbol{v}$ is the velocity vector, e is the internal energy, p is the thermodynamic pressure, θ is the thermodynamic temperature, $\boldsymbol{b}$ is a body force (typically, gravity), r is a heat source/sink term, ν is the kinematic viscosity coefficient, c_p is the specific heat at constant pressure, κ is the thermal diffusivity coefficient (in a fluid, thermal diffusion is typically assumed to be an isotropic process), $\nabla^s_{\boldsymbol{x}} = 1/2(\nabla + \nabla^T)$ is the symmetric part of the gradient, and $\boldsymbol{I} = \delta_{ij}$ is the identity (or Kronecker) tensor. In the case of a compressible fluid, density, thermodynamic pressure and internal energy are not independent of one another, but are related by an equation of state of the type

$$p = p(\rho, e) , \tag{11}$$

For most fluids, it is also possible to express the internal energy e in terms of the temperature θ as follows

$$e = c_v(\theta)\theta , \tag{12}$$

with c_v the specific heat at constant volume. Typically, c_v and c_p are functions of θ.

3 The origins of the SUPG method: Brooks and Hughes [5]

In 1982, Brooks and Hughes published the first journal article on the SUPG method [5], summarizing five years of work on the subject. At the time, a number of research groups in various academic institutions (see, e.g., Baba and Tabata [4], Tabata [75, 76, 77, 78]) were focusing their research on incorporating upwinding into finite element approximations, to enhance the stability of such methods in advection-dominated flow problems. The SUPG method is a residual-based upwinding technique (hence, variationally consistent), aimed at stabilizing Galerkin finite element methods based

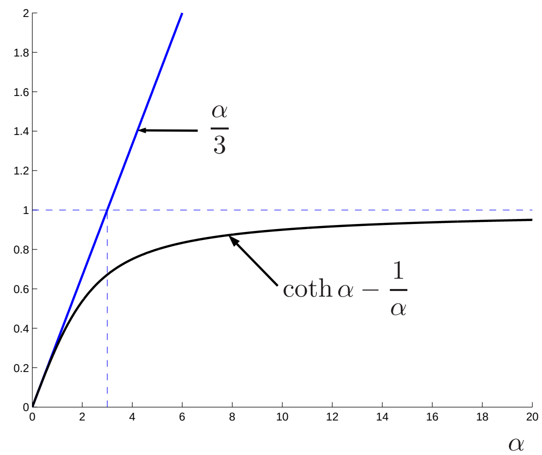

Figure 1: Behavior of $\tilde{\xi}(\alpha)$ defined in (17).

on equal-order interpolations. The effect of the SUPG term onto the equations is to transform the original Galerkin method into a *physics-adaptive* Petrov-Galerkin formulation. In Brooks and Hughes [5], applications to the scalar linear convection-diffusion problem and the incompressible Navier–Stokes equations were considered. For a semi-discrete variational formulation of the time-dependent, multi-dimensional, scalar advection-diffusion problem, the SUPG method reads

$$0 = \int_\Omega w^h \partial_t \phi^h + \int_\Omega \nabla w^h \cdot \left(-\boldsymbol{a}\phi^h + \kappa \nabla \phi^h \right) - \int_\Omega w^h f + \mathsf{SUPG}(w^h, \phi^h) \; , \tag{13}$$

with

$$\mathsf{SUPG}(w^h, \phi^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} p^e \left(\partial_t \phi^h + \boldsymbol{a} \cdot \nabla \phi^h - \kappa \Delta \phi^h - f \right) \; , \tag{14}$$

$$p^e = \tau^e \boldsymbol{a} \cdot \nabla w^h \; . \tag{15}$$

Here $\Omega = \bigcup_{e=1}^{n_{el}} \Omega_e$, and the Dirichlet boundary conditions are embedded in the test and trial spaces. The term p^e is called the *perturbation* to the test-function space, since it modifies the original Galerkin method into a Petrov-Galerkin method. The SUPG method is adaptive in the sense that it leverages the residual of the base Galerkin formulation to modify the structure of the variational formulation, and that the parameter τ^e (an intrinsic time scale) is chosen to adapt to both the convective and diffusive limit correctly (see, e.g., Fig. 1). In particular, the following definition of τ^e yields a *nodally exact solution* for the one-dimensional, steady case, for all Péclet numbers $\alpha = (||\boldsymbol{a}||_2 h)/(2\kappa)$:

$$\tau^e = \frac{2h_e}{||a||_2}\tilde{\xi}(\alpha) \ , \tag{16}$$

$$\tilde{\xi}(\alpha) = \left(\coth\alpha - \frac{1}{\alpha}\right) \ , \tag{17}$$

where h_e is the length of the eth element along the direction of advection (see [5] for a precise definition in the multi-dimensional case). Many alternative definitions of the parameter τ^e have been defined, sometimes of easier implementation in multiple dimensions [5, 35].

Johnson et al. [52] proved that the SUPG method is stable for all Péclet numbers, and that the order of convergence in the L^2-norm for the hyperbolic (pure advection) case is $p + 1/2$, where p is the order of the polynomial used in the finite element interpolation. Hence, the SUPG method is sub-optimal with gap 1/2 in the order of convergence. However, it was also observed in [52] that when the forcing term f is sufficiently smooth (case which encompasses the vast majority of practical applications), the numerically-observed order of convergence in the hyperbolic case is optimal (i.e., $p + 1$).

4 The first SUPG method for compressible flow: Hughes and Tezduyar [40, 91, 92]

Hughes and Tezduyar [40, 91, 92] made use of the quasi-linear form of Navier–Stokes equations to generalize the SUPG operator to compressible flow computations, with emphasis on the compressible Euler equations (i.e., $\mathsf{F}^{(d)} = \mathbf{0}$). We present the main discussion in this context, although the generalization to viscous compressible flows was already discussed in [40, 91, 92]. Let $\mathsf{F} = \mathsf{F}^{(c)} + \mathsf{F}^{(p)}$, and define by $\mathsf{F}_i^{(\cdot)}$ the ith column of the matrix $\mathsf{F}^{(\cdot)}$. Thus, the compressible Euler system reads (from now on, repeated index notation is implied unless otherwise stated):

$$\partial_t \mathsf{U} + \partial_{x_i}\mathsf{F}_i + \mathsf{G} = \mathbf{0} \ . \tag{18}$$

The quasi-linear advective form of (18), in terms of the vector U of conservation variables, is

$$\partial_t \mathsf{U} + \mathsf{A}_i(\mathsf{U})\,\partial_{x_i}\mathsf{U} + \mathsf{G} = \mathbf{0} \ , \tag{19}$$

where $\mathbf{A}_i = \nabla_{\mathbf{U}} \mathbf{F}_i$ is a $(n_d+2)\times(n_d+2)$-matrix. The matrices $\mathbf{A}_i$'s represent generalized advection, a combination of convective and acoustic effects. Hughes and Tezduyar [40, 91, 92] proposed a stabilized semi-discrete weak form of (1)–(4):

$$0 = \int_\Omega \left(\mathbf{W}^h \cdot \left(\partial_t \mathbf{U}^h + \partial_{x_i} \mathbf{F}_i^h + \mathbf{G}^h \right) \right) + \mathsf{SUPG}(\mathbf{W}^h, \mathbf{U}^h) \;, \qquad (20)$$

where the Dirichlet boundary conditions of type (2) are embedded in the definition of the function spaces, the superscript h indicates a discrete approximate, and

$$\mathsf{SUPG}(\mathbf{W}^h, \mathbf{U}^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \mathbf{P}(\mathbf{W}^h)^T \left(\partial_t \mathbf{U}^h + \partial_{x_i} \mathbf{F}_i^h + \mathbf{G}^h \right) \;, \qquad (21)$$

$$\mathbf{P}(\mathbf{W}^h) = \mathbf{T}_i^h \partial_{x_i} \mathbf{W}^h \;, \qquad (22)$$

$$\mathbf{T}_i^h = \tau_i^e \; \mathbf{A}_i^T(\mathbf{U}^h) \qquad \text{(no sum)} \;. \qquad (23)$$

Remark 1 The proposed method was indeed globally conservative in view of the application of a "group finite element" approach (see also Christie et al. [11], Fletcher [15], Spradley et al. [73]), such that

$$\mathbf{F}_i^h(\boldsymbol{x}, t) = \sum_B N_B(\boldsymbol{x}) \mathbf{F}_{i;B}(t) \;, \qquad (24)$$

where $\mathbf{F}_{i;B}(t)$ represents evaluation of the flux $\mathbf{F}_i^h$ at node B, with $N_B(\boldsymbol{x})$ the n_d-linear shape function (n_d is the number of space dimensions). This approximation of the fluxes is compatible with a discrete Gauss divergence theorem, and is therefore conservative.

The parameter τ_i^e was designed in [40, 91, 92] according to a temporal or a spatial criterion, similar to how this was done in [5]. The following definitions were tested:

$$\tau_i^e = c_\tau \Delta t \;, \qquad \text{(temporal criterion)} \;, \qquad (25)$$

$$\tau_i^e = c_\tau \frac{h}{a_\rho} \tilde{\xi} \;, \qquad \text{(spatial criterion \# 1)} \;, \qquad (26)$$

$$\tau_i^e = c_\tau \frac{h_i}{\rho(\mathbf{A}_i)} \tilde{\xi} \;, \quad \text{(no sum)} \qquad \text{(spatial criterion \# 2)} \;, \qquad (27)$$

where a_ρ is the discrete l^2-norm of the vector of components $\rho(\mathbf{A}_i)$, the spectral radii of the matrices $\mathbf{A}_i$. The term $\tilde{\xi}$, defined in (17), takes the value

1 for the compressible Euler equations. In addition, $h_i = 2||\nabla_{\boldsymbol{\xi}} x_i||$, with $\nabla_{\boldsymbol{\xi}}$ the gradient in the iso-parametric reference domain, and $h = h_i \rho(\mathbf{A}_i)/a_\rho$ (equal to zero if $a_\rho = 0$). The multiplicative constant c_τ was set equal to 1/2 (optimal choice) in most tests, but, due to the lack of a discontinuity capturing operator, it was observed in [40, 91, 92] that some test results would improve for different choices.

Remark 2 The fundamental idea in [40, 91, 92] is the use of the quasi-linear form (19) to interpret the matrices $\mathbf{A}_i$ in the context of generalized advection operators. This is the crucial step that opened an entire new field of application of the SUPG method. The choice $\mathbf{T}_i = \tau_i^e \mathbf{A}_i^T$ was justified in [40, 91, 92] as the one that leads to the correct transformation properties for the stabilization term, when diagonalization of the Euler system is possible, as in the one-dimensional case. This form for $\mathbf{T}_i$ is also consistent with the variational multi-scale analysis of the compressible flow equations (see Hauke and Hughes [25, 26], Hughes [28], Hughes et al. [29, 39], Scovazzi [69]).

Remark 3 Already in this early work, it was recognized the importance of designing the "τ" as a function of the time-step size, in the case of transient flows as indicated in (25). In later developments, the definition of τ for unsteady flows was derived using space-time concepts, in which the time and space axis are considered as generalized advection axes (see Section 8).

Remark 4 Various τs were proposed after those in [5] and [40, 91, 92], followed by the one introduced in [94], and those proposed in the subsequently reported SUPG-based methods. Defining a separate τ for each degree-of-freedom (i.e. for each equation, leading to a matrix form of the τ), was proposed in [33], and generalized in [72].

Hughes and Tezduyar [40, 91] also presented a thorough analysis of the effect of SUPG stabilization on stability and order of convergence, for a class of predictor/multi-corrector time integrators adopted in the computations. The numerical results showed good performance of the method in the case of steady shocks, and noisier results in the case of transient shocks. The results were in any case very encouraging, considering that no discontinuity capturing operator was applied, and the SUPG stabilization was required to control both linear and non-linear instabilities.

5 Entropy variables: Hughes et al. [31]

In the work of Hughes et al. [31, 32], a new formulation using entropy variables was presented. Similarly to Harten [22], the change from conservation to entropy variables is realized by constructing a *convex* entropy functional $\eta(\mathbf{U})$ with convex fluxes γ_i, satisfying $\nabla_{\mathsf{U}} \gamma_i = \nabla_{\mathsf{U}} \eta \; \mathbf{A}_i$. In view of (1), assuming for the moment $\mathbf{F}_i^{(d)} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$,

$$\partial_t \eta + \partial_{x_i} \gamma_i = \nabla_{\mathsf{U}} \eta \, (\partial_t \mathbf{U} + \mathbf{A}_i \partial_{x_i} \mathbf{U}) = \mathbf{0} \; . \tag{28}$$

Applying the Legendre transform to the entropy/entropy-flux pair, the symmetrizing transformation of variables $\mathbf{V} = (\nabla_{\mathsf{U}} \eta)^T$ can be obtained as the solution to a maximization problem (see Godunov [21], Moch [61]). Defining $\mathbf{F}_i^{(d)} = \mathbf{K}_{ij} \; \partial_{x_j} \mathbf{U}$ (where $\mathbf{K}_{ij}$ is symmetric positive semi-definite by definition), equation (1) can be reduced to

$$\tilde{\mathbf{A}}_0 \; \partial_t \mathbf{V} + \tilde{\mathbf{A}}_i \; \partial_{x_i} \mathbf{V} - \partial_{x_i} \left(\tilde{\mathbf{K}}_{ij} \; \partial_{x_j} \mathbf{V} \right) + \mathbf{G} = \mathbf{0} \; , \tag{29}$$

where $\tilde{\mathbf{A}}_0 = \nabla_{\mathsf{V}} \mathbf{U} = \nabla_{\mathsf{UU}} \eta$ and $\tilde{\mathbf{A}}_i = \mathbf{A}_i \tilde{\mathbf{A}}_0 = \nabla_{\mathsf{UU}} \gamma_i$, are symmmetric positive definite, and $\tilde{\mathbf{K}}_{ij} = \mathbf{K}_{ij} \tilde{\mathbf{A}}_0$ is symmetric positive semi-definite. Hughes et al. [31, 32] used the physical entropy variables, namely

$$\mathbf{V} = (\nabla_{\mathsf{U}} \eta)^T = \frac{1}{\rho e} \left\{ \begin{array}{c} -\mathsf{U}_5 + \rho e(\gamma + 1 - s) \\ \mathsf{U}_2 \\ \mathsf{U}_3 \\ \mathsf{U}_4 \\ -\mathsf{U}_1 \end{array} \right\} \tag{30}$$

where

$$\eta = \; - \rho s \; , \tag{31}$$

$$s = \; \ln \left(\frac{(\gamma - 1)\rho e}{\mathsf{U}_1^{\gamma}} \right) \; , \tag{32}$$

$$\rho e = \mathsf{U}_5 - \frac{\mathsf{U}_i \mathsf{U}_i}{2\mathsf{U}_1} \; , \tag{33}$$

with γ the isotropic constant for ideal gases. The importance of the development of a SUPG formulation for symmetric system was twofold:

1. On the one hand, it was shown by Hughes et al. [31, 32], testing the weak Galerkin formulation of the Navier–Stokes equations against $\mathbf{V}$, that a Galerkin method with symmetric variables embeds the

Clausius-Duhem entropy inequality. In the case of the Euler equations, Hughes et al. [31, 32] proved that the base Galerkin method embeds only an entropy equality, and entropy is conserved (this is evident upon multiplication of (28) by the admissible test function $\nabla_{\mathsf{U}} \eta = \mathbf{V}^T$, and integration over Ω). This result led to the understanding that, in the presence of shocks, additional dissipative mechanisms (possibly) in the form of artificial viscosities were needed.

2. On the other hand, Johnson and Szepessy [53], Johnson et al. [54], Szepessy [74], and Hughes et al. [32] were able to prove convergence to entropy solutions of symmetric systems of conservation laws, when the SUPG method was augmented with a discontinuity-capturing viscosity.

Remark 5 The development of stabilized methods with entropy variables helped understanding many fundamental aspects of SUPG methods, of great importance also for other sets of variables. Le Beau and Tezduyar [56] and Le Beau et al. [55] showed in a large number of tests that entropy and conservation variables, complemented by appropriate discontinuity capturing viscosities, yield virtually indistinguishable numerical solutions, greatly improved compared to what was reported in [40, 91, 92]. We will return to this point when discussing the later work of Hauke and Hughes [25, 26], in Section 10.

6 Developments in the design of the SUPG operator: Hughes and Mallet [33]

Using the symmetric form of the Navier–Stokes equations, Hughes and Mallet [33] developed a design paradigm for the SUPG operator alternative to [40, 91, 92]. The work of Hughes and Mallet for symmetric variables stems as a generalization of their joint work with Mizukami [35], for the case of the scalar advection-diffusion problem (13). According to Hughes et al. [35], the definition of τ^e in (15) is replaced by

$$\tau^e = \frac{\tilde{\xi}(\alpha^e)}{||\boldsymbol{b}^e||_p} \ , \tag{34}$$

$$\boldsymbol{b}^e = (\boldsymbol{a} \cdot \nabla)\boldsymbol{\xi}^e \ , \tag{35}$$

$$\alpha^e = \frac{||\boldsymbol{a}||_2^2}{\kappa||\boldsymbol{b}^e||_p} \ , \tag{36}$$

where $\boldsymbol{\xi}$ is the coordinate in the isoparametric *parent* domain, $\tilde{\xi}$ is defined analogously to (17), and $p = 2$ is a typical choice for the norms in the previous definitions. Note that the vector $\boldsymbol{b}^e$ collapses to $2a/h$ when $\boldsymbol{a}$ is aligned with the edge of length h of an element of square or cubic shape. Hughes and Mallet adopted diagonalization techniques to extend this concept to symmetric systems of conservation laws. To stabilize the semi-discrete weak form

$$0 = \int_\Omega \mathbf{W}^h \cdot \left(\tilde{\mathbf{A}}_0(\mathbf{V}^h)\, \partial_t \mathbf{V}^h + \tilde{\mathbf{A}}_i(\mathbf{V}^h)\, \partial_{x_i} \mathbf{V}^h + \mathbf{G}^h \right) + \mathsf{SUPG}(\mathbf{W}^h, \mathbf{V}^h)\ , \tag{37}$$

the design of the SUPG operator was developed in two steps (again, for simplicity, and without lack of generality, we consider only the Euler system): First, the one-dimensional case for a system of conservation laws was considered, and then, a generalization to multiple dimensions was sought. The quite involved details of the simultaneous diagonalization of the matrices $\tilde{\mathbf{A}}_0$, $\tilde{\mathbf{A}}_i$, and $\tilde{\mathbf{K}}_{ij}$ can be found in [33], and are not reported here, for the sake of brevity. In the purely hyperbolic case, however, the use of Cayley-Hamilton theorem allows to by-pass the eigenvalue problem, and to obtain a simple expression for the stabilization parameter:

$$\mathsf{SUPG}(\mathbf{W}^h, \mathbf{V}^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left(\tilde{\boldsymbol{\tau}}^e \tilde{\mathbf{A}}_i \partial_{x_i} \mathbf{W}^h \right)^T \left(\tilde{\mathbf{A}}_0\ \partial_t \mathbf{V}^h + \tilde{\mathbf{A}}_i\ \partial_{x_i} \mathbf{V}^h + \mathbf{G}^h \right) \tag{38}$$

$$\tilde{\boldsymbol{\tau}}^e = \tilde{\mathbf{A}}_0^{-1} \left(\sum_{i=1}^{n_d} \mathbf{B}_i^2 \right)^{-1/2} \tag{39}$$

where $\mathbf{B}_i = \partial \xi_i / \partial x_j \tilde{\mathbf{A}}_j(\mathbf{V}^h)$ plays a similar role to the the term $\boldsymbol{b}^e$ in (35), and $\tilde{\boldsymbol{\tau}}^e$ is a symmetric matrix.

Remark 6 Many subsequent designs of the stabilization matrix were profoundly influenced by the work of Hughes and Mallet, and can be thought of as generalizations and extensions of such methodology. As an example, the reader is prompted to compare (39) with the definition (55) of $\tilde{\boldsymbol{\tau}}$, proposed by Shakib et al. [72], or the one proposed by Hauke and Hughes [25, 26].

7 Discontinuity capturing operators

The studies on entropy variables showed the necessity of additional dissipation mechanisms when shock waves form in compressible flows. The development of discontinuity capturing operators was then undertaken, by observing that oscillations may be present when stabilized methods are applied to linear advection-diffusion problems, near sharp boundary layers or internal interfaces. The latter, in the context of the compressible Euler equations, are termed contact discontinuities (or equivalently, *linear waves*). Hughes et al. [35] proposed to use the projector in the direction of the solution gradient

$$\mathbf{\Pi} = \frac{\nabla \phi^h (\nabla \phi^h)^T}{||\nabla \phi^h||_2^2} \tag{40}$$

to augment the right hand side of equation (13) (considered, for the sake of simplicity, in the steady flow case) with

$$\mathsf{DC}(w^h, \phi^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\parallel}^e \boldsymbol{a}_{\parallel} \cdot \nabla w^h \left(\boldsymbol{a} \cdot \nabla \phi^h - \kappa \Delta \phi^h - f \right) \ , \tag{41}$$

where $\boldsymbol{a}_{\parallel}$ is the projection of $\boldsymbol{a}$ along $\nabla \phi^h$, that is,

$$\boldsymbol{a}_{\parallel} = \begin{cases} \mathbf{\Pi} \boldsymbol{a} \ , & \text{if } \nabla \phi^h \neq \mathbf{0} \ , \\ \mathbf{0} \ , & \text{if } \nabla \phi^h = \mathbf{0} \ , \end{cases} \tag{42}$$

and

$$\tau_{\parallel}^e = \frac{\tilde{\xi}(\alpha_{\parallel}^e)}{||\boldsymbol{b}_{\parallel}^e||_p} \ , \tag{43}$$

$$\boldsymbol{b}_{\parallel}^e = (\boldsymbol{a}_{\parallel} \cdot \nabla) \boldsymbol{\xi}^e \ , \tag{44}$$

$$\alpha_{\parallel}^e = \frac{||\boldsymbol{a}_{\parallel}||_2^2}{\kappa ||\boldsymbol{b}_{\parallel}^e||_p} \ . \tag{45}$$

Hughes and Mallet [34] generalized the projector $\mathbf{\Pi}$ to the symmetric form of the Navier–Stokes equations:

$$\tilde{\mathbf{\Pi}}_{ij} = \frac{\partial_{x_i} \mathbf{V}^h \ (\partial_{x_j} \mathbf{V}^h)^T \tilde{\mathbf{A}}_0(\mathbf{V}^h)}{\sum_{k,l=1}^{n_d} (\partial_{x_k} \mathbf{V}^h)^T \ \tilde{\mathbf{A}}_0(\mathbf{V}^h) \ \partial_{x_l} \mathbf{V}^h} \ , \tag{46}$$

where the matrix $\tilde{\mathbf{A}}_0$, used to scale correctly the projector, has the meaning of the *Riemannian metric*. Hence, (37) was augmented with the term

$$\mathsf{DC}(\mathbf{W}^h, \mathbf{V}^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left(\tilde{\boldsymbol{\tau}}_{\parallel}^e \tilde{\mathbf{A}}_{i;\parallel} \partial_{x_i} \mathbf{W}^h \right)^T \left(\tilde{\mathbf{A}}_0 \ \partial_t \mathbf{V}^h + \tilde{\mathbf{A}}_i \ \partial_{x_i} \mathbf{V}^h + \mathbf{G}^h \right) \ , \tag{47}$$

where, similarly to (42),

$$\tilde{\mathbf{A}}_{i;\|}^{T} = \tilde{\mathbf{A}}_{j}^{T} \tilde{\mathbf{\Pi}}_{ji} \ . \tag{48}$$

For the sake of brevity, we omit the definition of $\tilde{\boldsymbol{\tau}}_{\|}^{e}$, which is analogous to (43) and involves an eigenvalue/eigenvector problem.

Remark 7 Since the paper of Hughes and Mallet [34], the research aimed at improving discontinuity capturing operators has been quite intense. In particular, Shakib et al. [72] used similar generalized projection techniques in the context of a space-time formulation of the Galerkin/Least-Squares stabilized method of Hughes et al. [30] (for a detailed discussion, see Section 9). The work of Hauke and Hughes [25, 26] and Hauke [24] extended this type of discontinuity capturing operators to general sets of solution variables (see also Section 10).

Remark 8 A different approach to discontinuity capturing was presented by Le Beau and Tezduyar [56] and Le Beau et al. [55], and was more recently improved by Tezduyar et al. [85, 86, 95–97]. The much simplified approach in these references consists of defining a residual-based artificial viscosity ν_{DC}. Namely, for conservation variables,

$$\mathsf{DC}(\mathbf{W}^h, \mathbf{U}^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nu_{DC} \; \partial_{x_i} \mathbf{W}^h \cdot \partial_{x_i} \mathbf{U}^h \ . \tag{49}$$

A full discussion on such operators is presented in Section 11.2. A somewhat similar design approach was found particularly effective by Scovazzi et al. [70] and Scovazzi [69], for the highly-transient and very intense shock structures of Lagrangian hydrodynamics problems.

8 Space-time variational formulations

Hughes et al. [32] presented a space-time variational formulation for the compressible Navier–Stokes equations. The origin of the space-time formulation in the context of SUPG methods can be traced back to the work of Johnson et al. [52] for the unsteady case of scalar, linear, advection-diffusion problems. Consider a space-time domain $\mathsf{Q} = \Omega \times]0, T[\subset \mathbb{R}^{n_d} \times \mathbb{R}^+$ with lateral boundary $\mathsf{P} = \partial\Omega \times]0, T[$, as illustrated in the left-hand side of Figure 2. P is further divided into the Dirichlet portion of the lateral boundary P^g, and the Neumann portion of the lateral boundary P^h, such that $\mathsf{P}^g \cap \mathsf{P}^h = \emptyset$, and $\overline{\mathsf{P}^g \cup \mathsf{P}^h} = \mathsf{P}$. In order to derive a space-time numerical method, discrete test and trial function spaces have to be defined over time slabs $\mathsf{Q}_n = \Omega \times]t_{n-1}, t_n[$ such that $[0, T] = \overline{\bigcup_{n=1}^{N+1} \mathsf{Q}_n}$. This approach

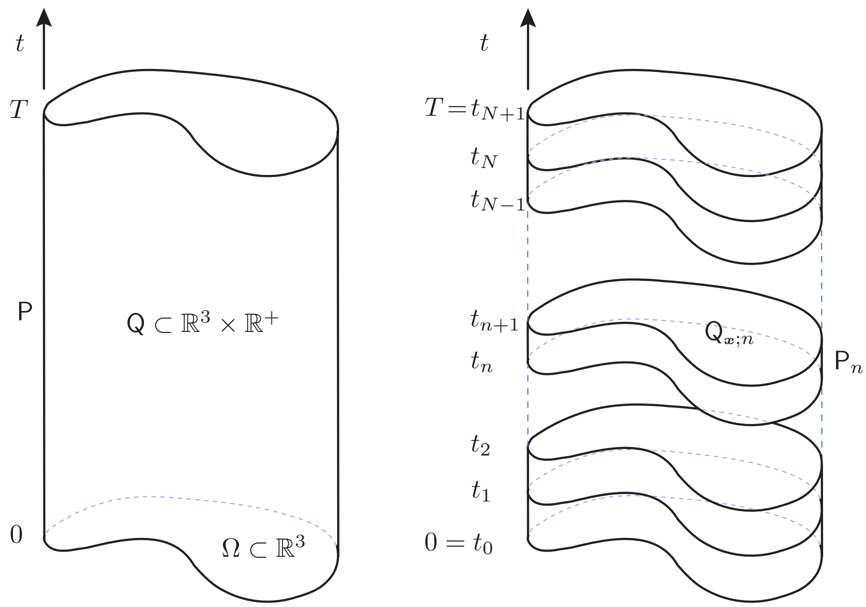

Figure 2: Space-time domain (left) and slicing into space-time slabs (right).

yields a time-stepping methodology, depicted on the right-hand side of Figure 2, which is *prismatic* in time, that is, each time slab Q_n is *extruded* from the domain Ω. Neumann boundary conditions are accounted for using integration by parts, while Dirichlet boundary conditions are imposed *strongly*. The space-time weak form of (1), in the case of symmetric entropy variables, reads:

$$
\begin{aligned}
0 = & \int_{\Omega} \mathbf{W}^h(t_{n+1}^-) \cdot \mathbf{U}(\mathbf{V}^h(t_{n+1}^-)) - \int_{\Omega} \mathbf{W}^h(t_n^+) \cdot \mathbf{U}(\mathbf{V}^h(t_n^-)) \\
& - \left(\int_{\mathsf{Q}_n} (\partial_t | \mathbf{W}^h) \cdot \mathbf{U}(\mathbf{V}^h) + \nabla \mathbf{W}^h : \mathbf{F}(\mathbf{V}^h) \right) + \int_{\mathsf{Q}_n} \mathbf{W}^h \cdot \mathbf{G}(\mathbf{V}^h) \\
& + \int_{\mathsf{P}_n^g} \mathbf{W}^h \cdot \Big(\mathbf{F}(\mathbf{V}^h) \boldsymbol{n} \Big) + \int_{\mathsf{P}_n^h} \mathbf{W}^h \cdot \mathbf{h} \\
& + \mathsf{SUPG}(\mathbf{W}^h, \mathbf{V}^h) + \mathsf{DC}(\mathbf{W}^h, \mathbf{V}^h) \ ,
\end{aligned}
\tag{50}
$$

where $\mathsf{DC}(\mathbf{W}^h, \mathbf{V}^h)$ is a discontinuity capturing operator to be defined subsequently, together with the SUPG operator. Also, $t^{\pm} = \lim_{\varepsilon \to 0^{\pm}} (t + \varepsilon)$, and $\mathbf{h}$ is the vector of Neumann boundary conditions. Imposing the Dirichlet boundary conditions strongly means that the corresponding entries of the vector $\mathbf{W}^h$ vanish at the Dirichlet boundary, where the boundary value of the solution is enforced.

Remark 9 (Euler-Lagrange equations) The Euler-Lagrange equations of the Galerkin part of the variational form (50) provide understanding of the nature of the variational formulation. They are obtained assuming that the solution is sufficiently smooth, so that integration by parts can be performed. If this is the case, (50) yields

$$
\begin{aligned}
0 = & \int_{\mathsf{Q}_n} \mathbf{W}^h \cdot \Big(\partial_t \mathbf{U}(\mathbf{V}^h) + \nabla \cdot \mathbf{F}(\mathbf{V}^h) + \mathbf{G}^h \Big) \\
& + \int_{\Omega} \mathbf{W}^h(t_n^+) \cdot [\![\mathbf{U}(\mathbf{V}^h(t_n))]\!] - \int_{\mathsf{P}_n^h} \mathbf{W}^h \cdot \Big(\mathbf{F}(\mathbf{V}^h) \boldsymbol{n}_{\boldsymbol{x}} - \mathbf{h} \Big) \ , \qquad (51)
\end{aligned}
$$

which enforces the Navier–Stokes equations on the interior of Q_n, Neumann boundary conditions on the boundary P_n^h, initial conditions at time t_n, through *causality* of the solution, that is, the *weak* continuity condition $[\![\mathbf{U}(\mathbf{V}^h(t_n))]\!] = \mathbf{U}(\mathbf{V}^h(t_n^+)) - \mathbf{U}(\mathbf{V}^h(t_n^-)) = \mathbf{0}$.

Remark 10 (Conservation properties) Global conservation of the formulation (50) is readily proved by choosing $\mathbf{W}^h$ constant over Q_n, assuming homogenous Neumann boundary conditions, and neglecting any source/sink terms ($\mathbf{G}^h = \mathbf{0}$). Hence (50) yields

$$
\int_{\Omega} \mathbf{U}(\mathbf{V}^h(t_{n+1}^-)) = \int_{\Omega} \mathbf{U}(\mathbf{V}^h(t_n^-)) \ , \qquad (52)
$$

which expresses global conservation between time t_{n+1}^- and t_n^-.

The first comprehensive study on space-time time integrators for the compressible Navier–Stokes was published in the work of Shakib and Hughes [71] and Shakib et al. [72], where first- and third-order time integrators were analyzed and applied to a number of compressible flow computations. The first-order time integrator was obtained using a discontinuous-in-time, piecewise-constant interpolation for both the test and trial space, while the third-order time integrator was obtained using discontinuous-in-time, piecewise-linear interpolation. In general, a discontinuous Galerkin method in time is of order $2k + 1$, where k is the order of the interpolation.

It is also worth mentioning the fact that Petrov-Galerkin space-time integrators are also available, in which discontinuous polynomials of order k are used for the test space (in time) and continuous polynomials of order $k+1$ are used for the trial space (again, in time). This choice leads to methods of accuracy $2k$, and dates back to Hulme [42], Jamet [43], and Aziz and Monk [3]. More recently, Johnson [51], French [17, 18], French and

Jensen [19], Estep and French [14], and French and Peterson [20] revived the interest in such integrators. The most common example is the method obtained in the case $k = 1$, which resembles a mid-point integrator, and leads to the Crank-Nocholson method for linear problems. This approach was successfully adopted by Scovazzi et al. [70] for explicit stabilized computations of Lagrangian shock-hydrodynamic flows and by Hoffman and Johnson [27] in adaptive Eulerian compressible flow computations.

9 The Galerkin/least-squares method

Hughes et al. [30] developed the Galerkin least/squares (GLS) stabilization method as a generalization of the SUPG approach, and applied it to scalar advection-diffusion problems in multiple dimensions and systems of symmetric conservation laws. Shakib et al. [72] applied GLS to a space-time formulation of the compressible Navier–Stokes equations, augmenting the right hand side of (52) with

$$\mathsf{SUPG}(\mathbf{W}^h, \mathbf{V}) = \sum_{e=1}^{n_{el}} \int_{\mathsf{Q}^e} (\tilde{\mathcal{L}} \mathbf{W}^h) \cdot \tilde{\boldsymbol{\tau}}^e (\tilde{\mathcal{L}} \mathbf{V}^h) \tag{53}$$

$$\tilde{\mathcal{L}} = \tilde{\mathbf{A}}_0 \, \partial_t \; + \tilde{\mathbf{A}}_i \, \partial_{x_i} \; - \partial_{x_i} \Big(\tilde{\mathbf{K}}_{ij} \, \partial_{x_j} \Big) + \tilde{\mathbf{C}} \tag{54}$$

$$\tilde{\boldsymbol{\tau}}^e = \tilde{\mathbf{A}}_0^{-1} \Bigg(\tilde{\mathbf{C}}^2 + \left(\frac{\partial \xi_0}{\partial x_0} \right)^2 \mathbf{I} + \left(\frac{\partial \xi_i}{\partial x_j} \frac{\partial \xi_i}{\partial x_k} \right) \tilde{\mathbf{A}}_j \tilde{\mathbf{A}}_k + \left(\frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_l} \frac{\partial \xi_j}{\partial x_m} \frac{\partial \xi_i}{\partial x_n} \right) \tilde{\mathbf{K}}_{kl} \tilde{\mathbf{K}}_{mn} \Bigg)^{(-1/2)} , \tag{55}$$

and

$$\mathsf{DC}(\mathbf{W}^h, \mathbf{V}^h) = \sum_{e=1}^{n_{el}} \int_{\mathsf{Q}^e} \nu_{DC} (\nabla_{[t,\boldsymbol{\xi}]} \mathbf{W}^h) \cdot \mathrm{diag}[\tilde{\mathbf{A}}_0] (\nabla_{[t,\boldsymbol{\xi}]} \mathbf{V}^h) \, , \tag{56}$$

where $\mathbf{I}$ is the identity matrix, $\boldsymbol{\xi}$ is the coordinate in the element's parent domain, $\mathbf{G}^h = \tilde{\mathbf{C}} \mathbf{V}^h$, $\mathrm{diag}[\tilde{\mathbf{A}}_0]$ is a block-diagonal matrix in which the block $\tilde{\mathbf{A}}_0$ is repeated as many times as necessary along the diagonal, and $\nabla_{[t,\boldsymbol{\xi}]}$ is a space-time generalized gradient which involves zeroth/first derivatives in space and time, and second derivatives in space. For a precise definition of $\nabla_{[t,\boldsymbol{\xi}]}$, the reader can refer to Shakib et al. [72], while it suffices to say that the proposed discontinuity capturing operator is acting along the direction of the Navier–Stokes operator, interpreted as a generalized gradient.

Remark 11 The GLS method was an important step in the development of stabilized methods for compressible flows since the stabilization term can be proved to be strictly dissipative. This an important aspect, since the base Galerkin formulation augmented with the GLS term satisfies a discrete entropy inequality, even in the inviscid limit. At the same time, when strong shocks are present, it was found important to introduce an additional discontinuity capturing operator.

Remark 12 The expression for $\tilde{\boldsymbol{\tau}}^e$ is obtained with a methodology similar to [33] for (39), leveraging diagonalization of the Navier–Stokes system.

Remark 13 The artificial viscosity ν_{DC} is a function of appropriate norms of the residual, which yields a linear or quadratic dependence [72]. The proposed approach stems from and extends the ideas in [34].

10 Computations with primitive variables: Hauke and Hughes [25, 26]

During the mid-to-late 1990s, Hauke and Hughes developed compressible SUPG formulations on sets of variables other than conservation or entropy variables. These formulations were derived from formulations with entropy variables, applying an additional change of variables:

$$\hat{\mathbf{A}}_0 = \tilde{\mathbf{A}}_0 \nabla_{\mathbf{Y}} \mathbf{V} \ , \tag{57}$$

$$\hat{\mathbf{A}}_i = \tilde{\mathbf{A}}_i \nabla_{\mathbf{Y}} \mathbf{V} \ , \tag{58}$$

$$\hat{\mathbf{K}}_{ij} = \tilde{\mathbf{K}}_{ij} \nabla_{\mathbf{Y}} \mathbf{V} \ , \tag{59}$$

$$\hat{\boldsymbol{\tau}}^e = \nabla_{\mathbf{Y}} \mathbf{V} \ \tilde{\boldsymbol{\tau}}^e \ . \tag{60}$$

All the previous matrices are in general non-symmetric. Hauke and Hughes [26] compared entropy and conservation variables with density primitive variables $\hat{\mathbf{Y}} = [\rho,\ \boldsymbol{v}^T,\ \theta]^T$, and pressure primitive variables $\hat{\mathbf{Y}} = [p,\ \boldsymbol{v}^T,\ \theta]^T$. The numerical results showed no significant differences in the case of compressible flow computations. However, only the pressure primitive variables are applicable to computations in the incompressible limit, since the matrix Jacobians $\hat{\mathbf{A}}_0$, $\hat{\mathbf{A}}_i$, and $\hat{\mathbf{K}}_{ij}$ stay bounded as the speed of sound tends to infinity.

Remark 14 Hauke and Hughes [25, 26] showed that the pressure primitive variables were the pathway to design a generalized stabilized method, for computations at all Mach numbers. In this context, it is important to mention the contribution of Wong et al. [100], who proposed a scaling of the matrix $\hat{\boldsymbol{\tau}}^e$ with the Mach number, to avoid degradation of accuracy in the nearly incompressible limit.

The GLS/SUPG stabilization term documented in [25, 26] reads

$$\mathsf{SUPG}(\mathbf{W}^h, \hat{\mathbf{Y}}^h) = \sum_{e=1}^{n_{el}} \int_{\mathsf{Q}_n^e} \left(\hat{\mathcal{L}}^T \mathbf{W}^h\right) \cdot \hat{\boldsymbol{\tau}}^e (\hat{\mathcal{L}} \hat{\mathbf{Y}}^h) \;, \tag{61}$$

with

$$\hat{\mathcal{L}} = \hat{\mathbf{A}}_0(\hat{\mathbf{Y}}^h)\partial_t + \hat{\mathbf{A}}_i(\hat{\mathbf{Y}}^h)\partial_{x_i} - \partial_{x_i}\left(\hat{\mathbf{K}}_{ij}(\hat{\mathbf{Y}}^h)\partial_{x_j}\right) + \hat{\mathbf{C}} \;, \tag{62}$$

$$\hat{\mathcal{L}}^T = \hat{\mathbf{A}}_0^T(\hat{\mathbf{Y}}^h)\partial_t + \hat{\mathbf{A}}_i^T(\hat{\mathbf{Y}}^h)\partial_{x_i} - \partial_{x_i}\left(\hat{\mathbf{K}}_{ij}^T(\hat{\mathbf{Y}}^h)\partial_{x_j}\right) + \hat{\mathbf{C}}^T \;, \tag{63}$$

where $\mathbf{G}^h = \mathbf{C}\hat{\mathbf{Y}}^h$.

Remark 15 Another important contribution found in [25, 26] is the consistent definition of the stabilizing perturbation to the test function in the case of non-symmetric systems. This result is obtained using the transformation $\nabla_{\mathbf{Y}}\mathbf{V}$ from entropy to non-symmetric variables, and confirms the initial work of Tezduyar and Hughes [91, 92] and Hughes and Tezduyar [40] for conservation variables (see eq. (23)). This definition is consistent with the the multi-scale framework (see, e.g., Hughes [28], Hughes et al. [29, 39], Scovazzi [69]).

Remark 16 In [25, 26] a simpler discontinuity capturing term $\mathsf{DC}(\mathbf{W}^h, \hat{\mathbf{Y}}^h)$ was used, namely

$$\mathsf{DC}(\mathbf{W}^h, \hat{\mathbf{Y}}^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nu_{DC}\, g^{ij} (\partial_{x_i}\mathbf{W}^h) \cdot \hat{\mathbf{A}}_0(\hat{\mathbf{Y}}^h)\; \partial_{x_j}\hat{\mathbf{Y}}^h \;, \tag{64}$$

where

$$g^{ij} = \left[\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_k}{\partial x_j}\right]^{-1} \tag{65}$$

is the metric tensor. This definition proved robust in hypersonic computations performed by Chalot and Hughes [10].

More recently Hauke [24] developed simplified forms of the stabilization and discontinuity capturing operators in the case of non-symmetric variables.

11 Stabilization and shock-capturing parameters

11.1 Stabilization parameters

For referential convenience, the original SUPG formulation of compressible flows in conservation variables [40, 91, 92] will be called "$(SUPG)_{82}$". The set of τs introduced in [40, 91, 92] in conjunction with $(SUPG)_{82}$ will be called "τ_{82}". The τ definition introduced in [94] automatically yields lower values for higher-order elements. The τ used in [56] with $(SUPG)_{82}$ is a slightly modified version of τ_{82}. The shock-capturing parameter used in [56] (defined with an expression in conservation variables, derived from the shock-capturing parameter designed for entropy variables) will be called "δ_{91}" here. Subsequent minor modifications of τ_{82} took into account the interaction between the shock-capturing and the $(SUPG)_{82}$ terms in a fashion similar to how it was done in [94] for advection–diffusion–reaction equations. Until recently, all these slightly modified versions of τ_{82} have always been used with the same δ_{91}, and we will categorize them here all under the label "$\tau_{82\text{-MOD}}$".

More recently, τs which are applicable to higher-order elements were proposed in [16] in the context of advective-diffusive systems. Calculating the τs based on the element-level matrices and vectors was introduced in [93] in the context of the advection–diffusion equation and the Navier–Stokes equations of incompressible flows. These definitions are expressed in terms of the ratios of the norms of the matrices or vectors. They automatically take into account the local length scales, advection field and the element Reynolds number. Based on these definitions, a τ can be calculated for each element or for each degree-of-freedom of each element, or, as it was proposed in [83, 87], for each integration point of each element. It was proposed in [84, 93] that the stabilization parameters to be used in advancing the solution from time level n to $n+1$ (including the parameter embedded in a stabilization term that resembles a discontinuity-capturing term) should be evaluated at time level n (i.e. based on the flow field already computed for time level n). This way we are spared from another level of nonlinearity. The element-matrix-based τ definitions (and their degree-of-freedom versions) introduced in [93] were applied in [7] (and in [8]) to $(SUPG)_{82}$, supplemented with the shock-capturing term with δ_{91}.

Various options for calculating the stabilization parameters in the context of the $(SUPG)_{82}$ formulation were introduced in [85, 86]. In this section we describe those options. For this purpose, we first define the acoustic speed as c, and define the unit vector $\boldsymbol{j}$ as

$$\boldsymbol{j} = \frac{\nabla \rho^h}{\| \nabla \rho^h \|} \; . \qquad (66)$$

In computing τ_{SUGN1} (advection-dominated limit of the stabilization parameter we are starting to define) for each component of the test vector-function $\mathbf{W}^h$, the stabilization parameters ${\tau^\rho}_{\text{SUGN1}}$, ${\tau^u}_{\text{SUGN1}}$ and ${\tau^e}_{\text{SUGN1}}$ (associated with the mass, momentum and energy balance equations) are defined by the following expression:

$$\tau^{\rho}{}_{\text{SUGN1}} = \tau^{u}{}_{\text{SUGN1}} = \tau^{e}{}_{\text{SUGN1}} = \left(\sum_{a=1}^{n_{en}} \left(c \left| \boldsymbol{j} \cdot \nabla N_a \right| + \left| \boldsymbol{v}^h \cdot \nabla N_a \right| \right) \right)^{-1} . \quad (67)$$

where n_{en} is the number of element nodes, $\boldsymbol{v}^h$ is the flow velocity, and N_a is the shape functions associated with node a. In computing τ_{SUGN2} (transient-dominated limit), the parameters $\tau^{\rho}{}_{\text{SUGN2}}$, $\tau^{u}{}_{\text{SUGN2}}$ and $\tau^{e}{}_{\text{SUGN2}}$ are defined as follows:

$$\tau^{\rho}{}_{\text{SUGN2}} = \tau^{u}{}_{\text{SUGN2}} = \tau^{e}{}_{\text{SUGN2}} = \frac{\Delta t}{2} . \quad (68)$$

In computing τ_{SUGN3} (diffusion-dominated limit), the parameter $\tau^{u}{}_{\text{SUGN3}}$ is defined by using the expression

$$\tau^{u}{}_{\text{SUGN3}} = \frac{h^2_{\text{RGN}}}{4\nu} , \quad (69)$$

where

$$h_{\text{RGN}} = 2 \left(\sum_{a=1}^{n_{en}} \left| \boldsymbol{r} \cdot \nabla N_a \right| \right)^{-1} , \qquad \boldsymbol{r} = \frac{\nabla \| \boldsymbol{v}^h \|}{\| \nabla \| \boldsymbol{v}^h \| \, \|} . \quad (70)$$

The parameter $\tau^{e}{}_{\text{SUGN3}}$ is defined as

$$\tau^{e}{}_{\text{SUGN3}} = \frac{\left(h^{e}{}_{\text{RGN}} \right)^2}{4\kappa^e} , \quad (71)$$

where

$$h^{e}{}_{\text{RGN}} = 2 \left(\sum_{a=1}^{n_{en}} \left| \boldsymbol{r}^e \cdot \nabla N_a \right| \right)^{-1} , \qquad \boldsymbol{r}^e = \frac{\nabla \theta^h}{\| \nabla \theta^h \|} . \quad (72)$$

The parameters $(\tau^{\rho}_{\text{SUPG}})_{\text{UGN}}$, $(\tau^{u}_{\text{SUPG}})_{\text{UGN}}$ and $(\tau^{e}_{\text{SUPG}})_{\text{UGN}}$ are calculated from their components by using the "*r-switch*" [93]:

$$(\tau^{\rho}_{\text{SUPG}})_{\text{UGN}} = \left(\frac{1}{(\tau^{\rho}{}_{\text{SUGN1}})^r} + \frac{1}{(\tau^{\rho}{}_{\text{SUGN2}})^r} \right)^{-\frac{1}{r}} , \quad (73)$$

$$(\tau^{u}_{\text{SUPG}})_{\text{UGN}} = \left(\frac{1}{(\tau^{u}{}_{\text{SUGN1}})^r} + \frac{1}{(\tau^{u}{}_{\text{SUGN2}})^r} + \frac{1}{(\tau^{u}{}_{\text{SUGN3}})^r} \right)^{-\frac{1}{r}} , \quad (74)$$

$$(\tau^{e}_{\text{SUPG}})_{\text{UGN}} = \left(\frac{1}{(\tau^{e}{}_{\text{SUGN1}})^r} + \frac{1}{(\tau^{e}{}_{\text{SUGN2}})^r} + \frac{1}{(\tau^{e}{}_{\text{SUGN3}})^r} \right)^{-\frac{1}{r}} , \quad (75)$$

where, typically, $r = 2$.

11.2 New shock-capturing technology and the $YZ\beta$ approach

In the context of shock-capturing, the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization was introduced in [82, 84] for incompressible flows with sharp gradients. The DCDD takes effect where there is a sharp gradient in the velocity field and introduces dissipation in the direction of that gradient. The way the DCDD is added to the formulation precludes augmentation of the SUPG effect by the DCDD effect when the advection and discontinuity directions coincide. The DCDD involves a second element length scale, which was also introduced in [82, 84] and is based on the solution gradient. This new element length scale is used together with the element length scales defined earlier in [94]. Recognizing this second element length as a diffusion length scale, new stabilization parameters for the diffusive limit were introduced for incompressible flows in [84, 86]. Partly based on the ideas underlying the new τs for incompressible flows, new ways of calculating the τs for compressible flows were introduced in [85, 86], and these new stabilization parameters were reviewed in Section 11.1. More significantly, new ways of calculating the shock-capturing parameters for compressible flows were also introduced in [86]. The objective was to have shock-capturing parameters that are simpler, and less costly to compute with, than δ_{91}. Some versions of these new shock-capturing parameters are based on ideas underlying the DCDD. Other versions, which were categorized as "$YZ\beta$ Shock-Capturing", are based on scaled residuals and are defined with options for smoother or sharper shocks. This approach is described next.

First, the "shock-capturing viscosity" ν_{SHOC} is defined as

$$\nu_{\text{SHOC}} = \left\| \mathbf{Y}^{-1} \mathbf{Z} \right\| \left(\sum_{i=1}^{n_d} \left\| \mathbf{Y}^{-1} \partial_{x_i} \mathbf{U}^h \right\|^2 \right)^{\beta/2-1} \left(\frac{h_{\text{SHOC}}}{2} \right)^{\beta}, \tag{76}$$

where $\mathbf{Y}$ is a diagonal scaling matrix constructed from the reference values of the components of $\mathbf{U}$:

$$\mathbf{Y} = \begin{bmatrix} (\mathsf{U}_1)_{\text{ref}} & 0 & 0 & 0 & 0 \\ 0 & (\mathsf{U}_2)_{\text{ref}} & 0 & 0 & 0 \\ 0 & 0 & (\mathsf{U}_3)_{\text{ref}} & 0 & 0 \\ 0 & 0 & 0 & (\mathsf{U}_4)_{\text{ref}} & 0 \\ 0 & 0 & 0 & 0 & (\mathsf{U}_5)_{\text{ref}} \end{bmatrix}, \tag{77}$$

$$\mathbf{Z} = \partial_t \mathbf{U}^h + \mathbf{A}_i \partial_{x_i} \mathbf{U}^h \tag{78}$$

or

$$\mathbf{Z} = \mathbf{A}_i^h \partial_{x_i} \mathbf{U}^h \tag{79}$$

and

$$h_{\rm SHOC} = h_{\rm JGN} \ , \tag{80}$$

$$h_{\rm JGN} = 2 \left(\sum_{a=1}^{n_{en}} |\boldsymbol{j} \cdot \nabla N_a| \right)^{-1} . \tag{81}$$

The parameter β is set as $\beta = 1$ for smoother shocks and $\beta = 2$ for sharper shocks. In a variation of the expression given by Eq. (76), $\nu_{\rm SHOC}$ is defined by the following expression:

$$\nu_{\rm SHOC} = \left\| \mathbf{Y}^{-1} \mathbf{Z} \right\| \left(\sum_{i=1}^{n_d} \left\| \mathbf{Y}^{-1} \partial_{x_i} \mathbf{U}^h \right\|^2 \right)^{\beta/2-1} \left\| \mathbf{Y}^{-1} \mathbf{U}^h \right\|^{1-\beta} \left(\frac{h_{\rm SHOC}}{2} \right)^{\beta} . \tag{82}$$

The compromise between the $\beta = 1$ and $\beta = 2$ selections is defined as the following averaged expression for $\nu_{\rm SHOC}$:

$$\nu_{\rm SHOC} = \frac{1}{2} \Big(\left(\nu_{\rm SHOC}\right)_{\beta=1} + \left(\nu_{\rm SHOC}\right)_{\beta=2} \Big) . \tag{83}$$

Versions of $\nu_{\rm SHOC}$ that take into account the Mach number and shock intensity across a shock was proposed in [96, 97]. In that, $\nu_{\rm SHOC}$ given by Eqs. (76) and (82) are modified as follows:

$$\nu_{\rm SHOC} \ \leftarrow \ \nu_{\rm SHOC} \left(1 + \left(\frac{\parallel \nabla \rho^h \parallel h_{\rm SHOC}}{\rho_{\rm ref}} \right) < M^{1/b_{\rm M}} - 1 > \right)^{2/b_{\rm F}} , \tag{84}$$

where M is the Mach number and "$< \cdots >$" is the Macaulay bracket:

$$< x - y > \ = \ \begin{cases} 0, & x \leq y \\ x - y, & x > y \end{cases} . \tag{85}$$

The reference density $\rho_{\rm ref}$ is defined as

$$\rho_{\rm ref} = \rho_{\rm inf} \left(\frac{\rho_{\rm sca}}{\rho_{\rm inf}} \right)^{b_{\rm R}/2} , \tag{86}$$

where $\rho_{\rm inf}$ is the density at the inflow and $\rho_{\rm sca}$ is a scaling density. In defining $\rho_{\rm sca}$, one of the options we consider is $\rho_{\rm sca} = \rho_{\rm inf}$. For flows with shocks, we also consider the options $\rho_{\rm sca} = \rho_2$ and $\rho_{\rm sca} = \rho_2 - \rho_1$, where ρ_1 and ρ_2 are the density values before and after a normal shock corresponding to the inflow Mach number. The parameters $b_{\rm M}$, $b_{\rm F}$ and $b_{\rm R}$ can each be set to 1 for smoother shocks and 2 for sharper shocks. Eq. (84), without the exponent $2/b_{\rm F}$, was originally introduced in [96]. With this expression, the definition of the shock-capturing viscosity takes into account the Mach number and shock intensity across a shock. The shock intensity is represented by the term $\left(\frac{\| \nabla\rho^h \| \, h_{\text{SHOC}}}{\rho_{\text{ref}}}\right)$, which is a scaled measure of the jump in density. The Mach number is represented by the term $< M^{1/b_{\text{M}}} - 1 >$, which becomes active for $M > 1$.

Based on Eq. (76), a separate ν_{SHOC} can be calculated for each component of the test vector-function $\mathbf{W}^h$:

$$(\nu_{\text{SHOC}})_{\text{I}} = \left|\left(\mathbf{Y}^{-1}\mathbf{Z}\right)_{\text{I}}\right| \left(\sum_{i=1}^{n_d} \left|\left(\mathbf{Y}^{-1}\partial_{x_i}\mathbf{U}^h\right)_{\text{I}}\right|^2\right)^{\beta/2-1} \left(\frac{h_{\text{SHOC}}}{2}\right)^{\beta}, \quad I = 1, 2, \ldots n_d + 2. \quad (87)$$

Similarly, a separate ν_{SHOC} for each component of $\mathbf{W}^h$ can be calculated based on Eq. (82):

$$(\nu_{\text{SHOC}})_{\text{I}} = \left|\left(\mathbf{Y}^{-1}\mathbf{Z}\right)_{\text{I}}\right| \left(\sum_{i=1}^{n_d} \left|\left(\mathbf{Y}^{-1}\partial_{x_i}\mathbf{U}^h\right)_{\text{I}}\right|^2\right)^{\beta/2-1} \left|\left(\mathbf{Y}^{-1}\mathbf{U}^h\right)_{\text{I}}\right|^{1-\beta} \left(\frac{h_{\text{SHOC}}}{2}\right)^{\beta}, \quad I = 1, 2, \ldots n_d + 2. \quad (88)$$

Given ν_{SHOC}, the shock-capturing term is defined as

$$S_{\text{SHOC}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \nabla\mathbf{W}^h : \left(\boldsymbol{\kappa}_{\text{SHOC}} \cdot \nabla\mathbf{U}^h\right) d\Omega \, , \quad (89)$$

where $\boldsymbol{\kappa}_{\text{SHOC}}$ is defined as $\boldsymbol{\kappa}_{\text{SHOC}} = \nu_{\text{SHOC}} \, \mathbf{I}$. As a possible alternative, it is defined as $\boldsymbol{\kappa}_{\text{SHOC}} = \nu_{\text{SHOC}} \, \boldsymbol{jj}$. If the option given by Eq. (87) or Eq. (88) is exercised, then ν_{SHOC} becomes an $(n_d + 2) \times (n_d + 2)$ diagonal matrix, and the matrix $\boldsymbol{\kappa}_{\text{SHOC}}$ becomes augmented from an $n_d \times n_d$ matrix to an $(n_d \times (n_d + 2)) \times ((n_d + 2) \times n_d)$ matrix.

To preclude compounding, ν_{SHOC} can be modified as follows:

$$\nu_{\text{SHOC}} \;\leftarrow\; \nu_{\text{SHOC}} - switch\left(\tau_{\text{SUPG}} \, \left(\boldsymbol{j} \cdot \boldsymbol{v}^h\right)^2, \; \tau_{\text{SUPG}} \, \left(|\boldsymbol{j} \cdot \boldsymbol{v}^h| - c\right)^2, \; \nu_{\text{SHOC}}\right), \quad (90)$$

where the "*switch*" function is defined as the "*min*" function or as the "*r-switch*" used earlier in this section. For viscous flows, the above modification would be made separately with each of $\tau^{\rho}_{\text{SUPG}}$, τ^{u}_{SUPG} and τ^{e}_{SUPG}, and this would result in ν_{SHOC} becoming a diagonal matrix even if the option given by Eq. (87) or Eq. (88) is not exercised.

A preliminary set of test computations with these new shock-capturing parameters were reported in [95] for inviscid supersonic flows. Those computations were limited to very simple 2D geometries and quadrilateral elements. A more comprehensive set of 2D test computations for inviscid supersonic flows were reported in [96]. Those tests with the $YZ\beta$ shock-capturing involved different element types and mesh orientations. In [97], numerical experiments were carried out for inviscid supersonic flows around cylinders and spheres to evaluate the performance of the $YZ\beta$ shock-capturing in more challenging test problems. In those numerical experiments, in addition to comparing the $YZ\beta$ results to those obtained with δ_{91}, for 2D structured meshes, the $YZ\beta$ result were compared to the results obtained with the OVERFLOW code [6]. All these test computations showed that, these new shock-capturing parameters are not only much simpler than δ_{91}, but also superior in accuracy.

In [66], the $YZ\beta$ shock-capturing was used in combination with the Variable Subgrid Scale (V-SGS) method, which was formulated for compressible flows in conservation variables in [65]. The V-SGS method was first introduced in [12] for the advection–diffusion–reaction equation and for incompressible flows. It is based on an approximation of the class of SGS models derived from the Hughes Variational Multiscale (Hughes-VMS) method [28]. The results reported in [66] show that the $YZ\beta$ shock-capturing yields better performance also when it is used in conjunction with the V-SGS method.

12 Stabilized ALE methods

Stabilized methods for compressible flows on arbitrary Lagrangian-Eulerian (ALE) meshes were initially developed by Masud [58], Rifai et al. [62, 63], for various engineering applications, among which fluid/structure interaction problems. These methods are easily implemented by modifying the advective flux $\mathsf{F}^{(c)}$ in (7) as follows:

$$\mathsf{F}^{(c)} = \mathsf{U} \otimes \boldsymbol{c} \,, \tag{91}$$

where $\boldsymbol{c} = \boldsymbol{v} - \hat{\boldsymbol{v}}$ is the *convective velocity* across the moving mesh and $\hat{\boldsymbol{v}}$ is the mesh velocity. The stabilization operators have to be modified accordingly.

Remark 17 The ALE framework is useful for the design of stabilized methods in their most general form, and was recently adopted by Scovazzi [67] and Scovazzi [68] to study the Galilean invariance properties of SUPG operators.

13 Stabilized Lagrangian methods

Scovazzi et al. [70] and Scovazzi [69] successfully developed SUPG-stabilized methods for compressible hydrodynamics computations (characterized by highly unsteady flows with Mach numbers in excess of 10^3) on Lagrangian meshes (i.e., $\boldsymbol{c} = \mathbf{0}$). Algorithms of shock hydrodynamics (*hydrocodes* in short) are traditionally developed on quadrilaterals/hexahedral meshes and, because of the piecewise-constant discretization of the thermodynamic fields, have never been successfully generalized to triangular/tetrahedral meshes. The advantage of using simplex-type meshes is evident in terms of automatic mesh generation, multi-material interface reconstruction, multi-physics radiation-hydrodynamics applications.

The methodology developed in [70] represented the first example of accurate and robust computations on simplex-type meshes of shock hydrodynamic flows, with comparable and sometimes superior quality to state-of-the-art hydrocodes on brick meshes. This approach is significantly different from mainstream stabilized methods for compressible flows, and the reader is prompted to see [69, 70] for specific details. A brief description of the main features is presented next:

1. The set of solution variables is given by the vector $\hat{\mathbf{Y}} = [\rho,\ \boldsymbol{v}^T,\ p]^T$.

2. An explicit predictor-multicorrector approach is adopted, since hydrocodes typically involve explicit time integration. The time integrator is the second-order space-time Petrov-Galerkin method described at the end of Section 8, and involves piecewise-linear, continuous trial functions in time, and piecewise-constant, discontinuous test functions in time. This methodology provides more compact storage and less computational burden with respect to the third-order algorithm proposed in [72].

3. To enhance performance and avoid either inverting matrices or solving eigenvalue/eigenvector problems on each element, a very simple diagonal matrix $\hat{\boldsymbol{\tau}}$ is used. Variational multi-scale interpretations are embedded in the proposed design for $\hat{\boldsymbol{\tau}}$ [69]. This simple appraoch performed very well in highly transient flows.

4. The discontinuity operator was designed as an isotropic Laplace operator acting on the current configuration gradients of the solution. This property was found very important in the solution of highly transient shock waves. The structure of the artificial-viscosity/discontinuity-capturing operator is isotropic in space, and somewhat similar to the $YZ\beta$ approach and (49).

5. In addition, and differently from any other discontinuity capturing approach in SUPG methods, the work done by the artificial stresses was introduced in the energy equation: This term was found crucial in the solution of piston-driven shock-wave problems.

14 Compressible turbulence

Compressible turbulence computations using pressure primitive variables were studied by Jansen et al. [46] and Jansen [44], where $\kappa - \varepsilon$ turbulence models were used to provide a cost-effective computational procedure. In later work of Hughes et al. [36, 37, 38], the variational multi-scale paradigm was used to provide a large eddy simulation (LES) model of turbulence, and was later incorporated by Whiting et al. [99] in a stabilized finite element method for compressible flows. Jansen et al. [47] also developed generalized-α time integrators for compressible turbulence computations, with improved high-frequency dissipation.

At the same time, Corsini et al. [13], Rispoli et al. [64] developed a number of residual-based eddy viscosity LES models aimed at stabilized compressible computations of turbo-machinery flows. The key idea in this work is the realization that a residual-based eddy viscosity *dynamically* adjusts to the conditions of the turbulent flow, and switches off for smooth (laminar) flow.

15 Summary

We reviewed 25 years of work on stabilized methods for compressible flows. We presented a unified view by tracking over time the main ideas that influenced the field, and by showing how these ideas evolved within the different research groups, from the origins until present times.

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