ICES REPORT 11-34

November 2011

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Reference: Tan Bui-Thanh, Leszek Demkowicz, and Omar Ghattas, "A Unified Discontinuous Petrov-Galerkin Method and its Analysis for Friedrichs' Systems", ICES REPORT 11-34, The Institute for Computational Engineering and Sciences, The University of Texas at Austin, November 2011.

A UNIFIED DISCONTINUOUS PETROV-GALERKIN METHOD AND ITS ANALYSIS FOR FRIEDRICHS' SYSTEMS

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Abstract. We propose a unified discontinuous Petrov–Galerkin (DPG) framework for Friedrichs-like systems, which embrace a large class of elliptic, parabolic, and hyperbolic partial differential equations (PDEs). The well-posedness, i.e., existence, uniqueness, and stability, of the DPG solution is established on a single abstract DPG formulation, and three abstract DPG methods corresponding to three different, but equivalent, norms are devised. We then apply the single DPG framework to several linear(ized) PDEs including, but not limited to, scalar transport, Laplace, diffusion, convection-diffusion, convection-diffusion-reaction, linear(ized) continuum mechanics (e.g., linear(ized) elasticity, a version of linearized Navier-Stokes equations, and etc), time-domain acoustics, and a version of the Maxwell's equations. The results show that we not only recover several existing DPG methods, but also discover new DPG methods for both PDEs currently considered in the DPG community and new ones. As a direct consequence of the single abstract DPG framework, all the DPG methods are shown to be trivially well-posed.

Key words. discontinuous Petrov–Galerkin methods; well-posedness; partial differential equations; Friedrichs' systems; inf–sup condition; consistency; stability; convergence.

AMS subject classifications. 65N30, 65N12, 65N15, 65N22.

1. Introduction. The discontinuous Petrov–Galerkin (DPG) framework introduced by Demkowicz and Gopalakrishnan [9, 11] has been evolving as a new numerical methodology for partial differential equations (PDEs). The method has been successfully applied to a wide range of PDEs including scalar transport equations [6,9,11], Laplace equation [10], convection-diffusion equations [10,11], Helmholtz equations [12,14,26], Burger and Navier-Stokes equations [7], and linear elasticity [5]. The DPG framework starts by partitioning the domain of interest into non-overlapping elements. Variational formulations are posed for each element separately and then summed up to form a global variational statement. Elemental solutions are connected by introducing hybrid variables (also known as fluxes or traces) that live on the skeleton of the mesh. This is therefore a mesh-dependent variational approach in which both bilinear and linear forms depend on the mesh under consideration.

In general, the trial and test spaces are not related to each other. In the standard Bubnov–Galerkin (also known as Galerkin) approach, the trial and test spaces are identical, while they differ in a Petrov–Galerkin scheme. Traditionally, one chooses either Galerkin or Petrov–Galerkin approaches, then proves the consistency and stability in both infinite and finite dimensional settings (if possible). The DPG method introduces a new paradigm in which one selects both trial and test spaces at the same time to satisfy well-posedness. In particular, one can select trial and test function spaces for which the continuity and inf–sup constants are unity. Given a finite dimensional trial subspace, the finite dimensional test space is constructed in such a way that the well-posedness of the finite dimensional setting is automatically inherited from the infinite dimensional counterpart.

For example, the DPG method in [11] starts with a given norm in the trial space

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and then seeks a norm in the test space in order to achieve unity continuity and inf–sup constants. Another DPG method in [12] achieves the same goal but reverses the process, i.e., it looks for a norm in the trial space corresponding to a given norm in the test space. Clearly, this is one of the advantages of the DPG methodology, since it allows one to choose a norm of interest to work with, while rendering the error optimal, i.e., smallest in that norm. Furthermore, the DPG methodology provides a natural framework for constructing robust versions of the method for singular perturbation problems, enabling automatic adaptivity. We shall not discuss the advantages of the DPG methods any further here, and the readers are referred to the original DPG papers [9–12] for more details.

Perhaps, one of the most challenging problems that needs to be addressed in developing a DPG method is to establish the well-posedness of the DPG formulation on the infinite dimensional level, from which the well-posedness of a finite dimensional DPG approximation inherits. This has been investigated for DPG formulations of linear first order hyperbolic [6], Laplace [10], convection-diffusion [10], Helmholtz [14], and linear elasticity [5] equations. The methods of proof however vary from one type of PDE to another, though sharing some similarities. Consequently, it might prevent practitioners from applying the DPG methodology to a new PDE until its well-posedness is available. Otherwise, there is no guarantee that a DPG method would behave as designed in the original work of Demkowicz and Gopalakrishnan [9, 11].

Meanwhile, a unified analysis of Discontinuous Galerkin (DG) methods for elliptic/parabolic/hyperbolic PDEs and beyond has been devised in a series of papers by Ern and Guermond [18–20]. This is possible due to the recent revised theory of Friedrichs' system [22] in a Hilbert space setting [24], rigorously formalized and advanced by [21], and further advanced by [1–3]. Ern and Guermond [18–20] have been successful in recovering most of the existing DG methods and discovering new ones for various PDEs including transport, convection-diffusion-reaction, linear(ized) continuum mechanics, and Maxwell's equations, to name a few.

The success of Ern and Guermond [18–20] inspires and motivates us to develop a unified theory for the DPG methodology for a large class of PDEs, and this is the main focus of the paper. In particular, we review the theory of Friedrichs-like systems under a Hilbert space setting [21] in Section 2.1. We next develop a single abstract DPG framework, prove its well-posedness, and derive three abstract DPG methods corresponding to three different, but equivalent, norms in Section 2.2. It is then followed by the convergence analysis of ideal and practical DPG methods in Sections 2.3 and 2.4, respectively. Section 3 reviews Friedrichs' systems of first order PDEs, followed by Friedrichs' systems of first order PDEs with partial coercivity in Section 4. To show the effectiveness of the single abstract framework, Section 5 applies it to various PDEs including transport, convection-diffusion-reaction, linear(ized) continuum mechanics, time-domain acoustic, and a version of the Maxwell's equations. As will be shown, our unified framework not only recovers several existing DPG methods, but also discovers new DPG methods for both PDEs currently considered in the DPG community and new ones. More importantly, a single well-posedness proof established for the abstract and unified DPG methodology is carried over to all Friedrichs-like systems in general and to all PDEs considered in Section 5 in particular. Finally, Section 6 concludes the paper with future directions.

2. Abstract theory.

2.1. Theory of Friedrichs' systems in a Hilbert space setting. In this section, we briefly review important theoretical advances of Friedrichs' systems in

Hilbert space settings due to Ern, Guermond, and Caplain [21] that are useful for our later developments. To begin, let L be a real Hilbert space equipped with the inner product $(\cdot,\cdot)_L$ and the induced norm $\|\cdot\|_L$. We identify L with its dual L' by the Riesz representation theorem. Assume that we have two linear operators (possibly unbounded) $T : \mathscr{D} \to L$ and $\tilde{T} : \mathscr{D} \to L$ satisfying the following two properties:

$$(T\varphi,\psi)_L = \left(\varphi,\tilde{T}\psi\right)_L, \qquad \forall \varphi,\psi \in \mathscr{D}, \tag{2.1a}$$

$$\left\|\left(T+\tilde{T}\right)\varphi\right\|_L \le c\,\|\varphi\|_L, \qquad \forall \varphi \in \mathscr{D}, \tag{2.1b}$$

where $\mathscr{D}$ is a dense subspace of L.

The abstract theory presented in this section is general. However, to connect the theory with familiar mathematical objects and to carry the intuition along, one may think of L as the space of square integral (vector-valued) functions over an open and bounded domain $\Omega \subset \mathbb{R}^d$, i.e., $L^2(\Omega)$, $\mathscr{D}$ as $C_0^\infty(\Omega)$, and T as a differential operator with its formal adjoint $\tilde{T}$.

It is easy to see that $\mathscr{D}$ equipped with the scalar product $(\cdot,\cdot)_T = (\cdot,\cdot)_L + (T\cdot,T\cdot)_L$ is an inner product space whose completion is denoted by W_0. The induced norm $\|\cdot\|_T = \sqrt{(\cdot,\cdot)_L + (T\cdot,T\cdot)_L}$ is known as the graph norm. One can show that the completion of $\mathscr{D}$ with respect to $(\cdot,\cdot)_{\tilde{T}} = (\cdot,\cdot)_L + \left(\tilde{T}\cdot,\tilde{T}\cdot\right)_L$ coincides with W_0. As a direct consequence, $T,\tilde{T} : (\mathscr{D},\|\cdot\|_T) \to (L,\|\cdot\|_L)$ are linear and continuous, and hence they can be extended by density to linear and continuous operators (again denoted by T and $\tilde{T}$) $T,\tilde{T} : (W_0,\|\cdot\|_T) \to (L,\|\cdot\|_L)$. Also by density, (2.1) can be extended to be valid for all $\varphi,\psi \in W_0$. Moreover, it can be shown that the adjoints of T and $\tilde{T}$ are the unique extensions of $\tilde{T}$ and T, again denoted by $\tilde{T}$ and T such that $\tilde{T},T : L \to W_0'$ and

$$\langle Tu,v\rangle_{W_0'\times W_0} = \left(u,\tilde{T}v\right)_L, \quad \forall u \in L, v \in W_0,$$

$$\left\langle \tilde{T}u,v\right\rangle_{W_0'\times W_0} = (u,Tv)_L, \quad \forall u \in L, v \in W_0.$$

By density, (2.1b) is also valid for all $\varphi \in L$.

We are interested in the solvability of the problem

$$Tu = f \in L, \tag{2.2}$$

and its solutions generally belong to the following graph space

$$W = \{u \in L : Tu \in L\},$$

which can be shown to coincide with the dual graph space

$$\left\{v \in L : \tilde{T}v \in L\right\}.$$

It is not difficult to see that W is a Hilbert space when equipped with the graph inner product $(\cdot,\cdot)_W = (\cdot,\cdot)_T$. However, the graph space is too general to provide the well-posedness of (2.2) and our next step is to find a subspace $V \subseteq W$ such that $T : V \to L$ is isomorphism. We begin by defining the following *boundary operator*:

$$\langle Bu,v\rangle_{W'\times W} = (Tu,v)_L - \left(u,\tilde{T}v\right)_L, \quad \forall u,v \in W. \tag{2.3}$$

Then, one can show that $B \in \mathscr{L}(W, W')$, and B is self-adjoint [21].

Let us define two cones [21]

$$C^{\pm} = \left\{ w \in W : \pm \left\langle Bw, w \right\rangle_{W' \times W} \geq 0 \right\},$$

and assume that there exist $V, V^* \subset W$ such that

$$\begin{array}{ll} V \subset C^+, & V^* \subset C^- \\ V = B\left(V^*\right)^{\perp}, & V^* = B\left(V\right)^{\perp}. \end{array} \tag{2.4}$$

This cone formalism for V and V^* is more natural than the following definition using an extra adhoc operator M. Assume there exists $M \in \mathscr{L}(W, W')$ such that

$$\left\langle Mw, w \right\rangle \geq 0, \quad \forall w \in W, \tag{2.5a}$$

$$W = \mathcal{N}\left(B - M\right) + \mathcal{N}\left(B + M\right), \tag{2.5b}$$

with $\mathcal{N}$ denoting the nullspace of its argument, and define

$$V = \mathcal{N}\left(B - M\right), \quad V^* = \mathcal{N}\left(B + M^*\right).$$

In order to obtain the desired result in Theorem 2.1, we further assume that T and $\tilde{T}$ satisfy the following positiveness condition

$$\left(\left(T + \tilde{T}\right)\varphi, \varphi\right)_L \geq 2\mu_0 \left\|\varphi\right\|_L^2, \quad \forall \varphi \in \mathscr{D}, \quad \mu_0 > 0, \tag{2.6}$$

which, by density, also holds for all $\varphi \in L$.

Now, either using the cone formalism or using the boundary operator M to define V and V^* we have the following well-posedness result whose proof can be found in [21].

Theorem 2.1. *Both $T : V \to L$ and $\tilde{T} : V^* \to L$ are isomorphisms. Furthermore, given $f_1, f_2 \in L$, then the following problems are well-posed:*

i) Seek $u \in V$ such that $Tu = f_1$ in L,

ii) Seek $v \in V^*$ such that $\tilde{T}v = f_2$ in L.

In particular, if u and v are the solutions of i) and ii), respectively, then they satisfy the following stability estimates:

$$\left\|u\right\|_L \leq \frac{1}{\mu_0}\left\|f_1\right\|_L, \quad \left\|u\right\|_W \leq \left(1 + \frac{1}{\mu_0}\right)\left\|f_1\right\|_L,$$

$$\left\|v\right\|_L \leq \frac{1}{\mu_0}\left\|f_2\right\|_L, \quad \left\|v\right\|_W \leq \left(1 + \frac{1}{\mu_0}\right)\left\|f_2\right\|_L.$$

Before starting our theoretical development, we extract from [21] the following useful result on B and M.

Theorem 2.2. *There hold:*

$$W_0 = \mathcal{N}\left(B\right) = \mathcal{N}\left(M\right) = \mathcal{N}\left(M^*\right), \tag{2.7a}$$

$$W_0^{\perp} = \mathcal{R}\left(B\right) = \mathcal{R}\left(M\right) = \mathcal{R}\left(M^*\right), \tag{2.7b}$$

where $\mathcal{R}$ denotes the range space.

2.2. Abstract DPG formulation. We are interested in the following inhomogeneous problem:

$$\left\{ \begin{array}{l} \text{Given } g \in W, f \in L. \text{ Seek } u \in W \text{ such that} \\ Tu = f \text{ in } L, \text{ and} \\ (u-g) \in V = \mathcal{N}(B-M), \end{array} \right. \tag{2.8}$$

which is clearly well-posed by Theorem 2.1. A weak formulation of (2.8) can be obtained as follows. From the strong formulation (2.8), we have

$$\frac{1}{2}(Tu, v)_L + \frac{1}{2}(Tu, v)_L = (f, v)_L,$$

which becomes

$$(Tu, v)_L + \frac{1}{2}\left\langle (M-B)\, u, v \right\rangle_{W' \times W} = (f, v)_L + \frac{1}{2}\left\langle (M-B)\, g, v \right\rangle_{W' \times W},$$

where we have used (2.3) and $(u-g) \in V = \mathcal{N}(B-M)$. The weak formulation now reads

$$\left\{ \begin{array}{l} \text{Given } g \in W, f \in L. \text{ Seek } u \in W \text{ such that, } \forall v \in W, \\ (Tu, v)_L + \frac{1}{2}\left\langle (M-B)\, u, v \right\rangle_{W' \times W} = (f, v)_L + \frac{1}{2}\left\langle (M-B)\, g, v \right\rangle_{W' \times W}. \end{array} \right. \tag{2.9}$$

The following result shows that the strong formulation (2.8) and its weak counterpart (2.9) are equivalent.

Proposition 2.3. *$u \in W$ is the solution of (2.8) iff it is the solution of (2.9). In particular, the weak formulation (2.9) is well-posed.*

Proof. The following proof is a variant of the proof in [18] for homogeneous boundary condition. The necessity is clear. For the sufficiency, take $v \in W_0$ and apply Theorem 2.2 to conclude that $Tu = f$ in L by the density of W_0 in L. Consequently, (2.9) implies $(M-B)(u-g) = 0$ in W', and this ends the proof. □

Remark 2.4. *We do not use the intermediate result in Proposition 2.3 for the DPG method, but it is interesting in its own right as a weak formulation for discontinuous Galerkin methods [18]. Moreover, the factor $\frac{1}{2}$ can be replaced by an arbitrary number and the result still holds. However, it plays an important role in the derivation and analysis of the abstract DPG method that we pursue below, in particular, it is used in Lemma 2.5 and Proposition 2.9.*

Either the strong or the weak formulation is appealing since they avoid taking the trace of functions in W, which may not be well-defined in general. More importantly, they permit us to study the well-posedness of an abstract DPG method in a quite general setting. To begin, let us assume that L is a Hilbert space of functions on an open and bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. We partition the domain Ω into N^{el} non-overlapping elements $K_j, j = 1, \ldots, N^{\text{el}}$ with Lipschitz boundaries such that $\Omega_h = \cup_{j=1}^{N^{\text{el}}} K_j$ and $\overline{\Omega} = \overline{\Omega}_h$. Here, h is defined as $h = \max_{j \in \{1,\ldots,N^{\text{el}}\}} diam\,(K_j)$. As a result, all the above results (assumptions respectively) are valid (assumed respectively) elementwise. We will attach the domain under consideration to operators and spaces whenever it is necessary to avoid confusion.

Decompose the first term of the left side of (2.9) and use definition (2.3), we obtain

$$\sum_{j=1}^{N^{\text{el}}} \left(u, \tilde{T}v\right)_{L(K_j)} + \sum_{j=1}^{N^{\text{el}}} \left\langle B_{K_j} u, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\left\langle (M-B)\, u, v\right\rangle_{W'(\Omega)\times W(\Omega)}$$
$$= \sum_{j=1}^{N^{\text{el}}} (f, v)_{L(K_j)} + \frac{1}{2}\left\langle (M-B)\, g, v\right\rangle_{W'(\Omega)\times W(\Omega)},$$

where u appearing in the duality pairings in the second term of the left side is understood as the restriction of u on K_j.

Now, it is natural to seek u in $L(\Omega_h) = L(\Omega)$, but then definition (2.3) is no longer valid. Therefore, we define a new variable q and consider the following DPG formulation:

$$\text{Given } g \in W(\Omega), f \in W'(\Omega_h). \text{ Seek } (u, q) \in L(\Omega_h) \times \tilde{W}(\Omega) \text{ such that}$$
$$\sum_{j=1}^{N^{\text{el}}} \left(u, \tilde{T}v\right)_{L(K_j)} + \sum_{j=1}^{N^{\text{el}}} \left\langle B_{K_j} q, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\left\langle (M-B)\, q, v\right\rangle_{W'(\Omega_h)\times W(\Omega_h)}$$
$$= \sum_{j=1}^{N^{\text{el}}} \left\langle f, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\left\langle (M-B)\, g, v\right\rangle_{W'(\Omega_h)\times W(\Omega_h)}, \quad \forall v \in W(\Omega_h), \quad (2.10)$$

which is meaningful with the following explanations:

- We have used a version of the Hahn-Banach theorem [17] (or any other valid continuous extensions) to extend $(M-B)\, q$ and $(M-B)\, g$ from $W'(\Omega)$ to $W'(\Omega_h)$, again denoted by $(M-B)\, q$ and $(M-B)\, g$, respectively. Note that both extensions are, in general, not unique. We therefore impose the following compatibility condition:

$$\begin{gathered} (M-B)\, q = (M-B)\, g \text{ in } W'(\Omega) \\ \Downarrow \\ (M-B)\, q = (M-B)\, g \text{ in } W'(\Omega_h). \end{gathered} \quad (2.11)$$

  At this level of abstraction, the use of the Hahn-Banach extension argument together with the compatibility condition is necessary for our theory to be rigorous. In practice, both conditions are often trivially satisfied as demonstrated in all examples in this paper.
- We define $\tilde{W}(\Omega) = W/Q(\Omega)$ as the quotient space with Q given by

$$Q = \{q \in W(\Omega) : a(q, v) = 0, \forall v \in W(\Omega_h)\},$$

  where

$$a(q, v) = \sum_{j=1}^{N^{\text{el}}} \left\langle B_{K_j} q, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\left\langle (M-B)\, q, v\right\rangle_{W'(\Omega_h)\times W(\Omega_h)}.$$

  Clearly, Q is a closed subspace of $W(\Omega)$, and hence it is meaningful to define the norm in $\tilde{W}(\Omega)$ as

$$\|q\|_{\tilde{W}} = \inf_{r \in W(\Omega): r-q \in Q} \|r\|_W, \quad \forall q \in \tilde{W}(\Omega).$$

- It should be pointed out that we have relaxed f in the DPG formulation (2.10) so that it now lives in the dual space $W'(\Omega_h) \supset L(\Omega)$ of the broken graph space $W(\Omega_h)$.

For convenience, we equivalently write (2.10) in the usual form $b((u,q),v) = \ell(v)$, where the bilinear form $b((u,q),v)$ and the linear form $\ell(v)$ are obviously defined as the right and left sides of (2.10), respectively.

The first step is to study the consistency of our DPG formulation. That is, if the data is sufficiently smooth, the strong solution of (2.8) should be a solution of the DPG formulation and vice versa.

LEMMA 2.5 (Consistency). *Assume $f \in L(\Omega)$. If $u \in W(\Omega)$ is a solution of (2.8), then $(u,u) \in L(\Omega_h) \times \tilde{W}(\Omega)$ is a solution of the DPG equation (2.10). Conversely, if $(u,q) \in L(\Omega_h) \times \tilde{W}(\Omega)$ is a solution of (2.10), then u is a solution of (2.8).*

Proof. Let u be the unique solution of (2.8) and set $q = u$. Using the compatibility condition (2.11) and (2.3) we conclude that $(u,q) = (u, q = u)$ solves the DPG formulation (2.10).

Conversely, take $v = \varphi \in W_0(\Omega)$ in (2.10), then use Theorem 2.2 and (2.3) we have

$$(f,\varphi)_{L(\Omega)} = \left(u, \tilde{T}\varphi\right)_{L(\Omega)} = \langle Tu, \varphi\rangle_{W_0'(\Omega)\times W_0(\Omega)}, \quad \forall \varphi \in W_0(\Omega),$$

from which it follows that $Tu = f \in L(\Omega)$, i.e., $u \in W(\Omega)$. What remains to be done is to show that $(u-g) \in V = \mathcal{N}(B-M)$.

Using (2.3) and taking $v \in W(\Omega)$ the ultra weak formulation (2.10) becomes

$$\langle B(q-u), v\rangle_{W'(\Omega)\times W(\Omega)} = \frac{1}{2}\langle (M-B)(g-q), v\rangle_{W'(\Omega)\times W(\Omega)},$$

and hence

$$B(q-u) = (M-B)\frac{(g-q)}{2} \text{ in } W'(\Omega). \tag{2.12}$$

Now, given (2.5a), it can be shown that (2.5b) is equivalent to

$$W = \mathcal{N}(B - M^*) + \mathcal{N}(B + M^*),$$

which, after using a similar argument as in [21], implies

$$\overline{\mathcal{R}(B-M)} \cap \overline{\mathcal{R}(B+M)} = \{0\}.$$

Since $\mathcal{R}(B) = \mathcal{R}(M)$ as stated in Theorem 2.2, it follows that

$$\overline{\mathcal{R}(B-M)} \cap \overline{\mathcal{R}(B)} = \{0\}. \tag{2.13}$$

Combining (2.12) and (2.13) yields

$$(B-M)(u-g) = 0,$$

and hence u is a solution of (2.8). ∎

COROLLARY 2.6. *Assume $f \in L(\Omega)$. There exists a unique solution (u,q) for the DPG formulation (2.10). Furthermore, the component q of the solution satisfies the boundary condition, i.e., $(B-M)(q-g) = 0$.*

Proof. Lemma 2.5 indicates that there exists a solution (u, q) for the DPG formulation (2.10) and the first component u is unique since it solves the strong equation (2.8). To prove the uniqueness of q, we first assume that (u, q_1) and (u, q_2) are two solutions of (2.10). Then, a simple subtraction shows that $(q_1 - q_2) \in Q$, which in turns implies that $q_1 = q_2$ in the quotient space $\tilde{W}(\Omega)$. The last assertion is obvious from the last steps in the proof of Lemma 2.5. ▯

It should be pointed out that Corollary 2.6 provides the existence and uniqueness of the DPG solution for $f \in L(\Omega)$. In this case, the stability of the component u is ready due to the well-posedness of the strong problem (2.8). In order to obtain the well-posedness of the DPG formulation, the existence and uniqueness together with stability of both u and q must be established for all $f \in W'(\Omega_h)$. To this end, we define norms in trial and test spaces such that both continuity and inf-sup constants are unity. One way to construct such norms is via a simple application of the Cauchy-Schwarz inequality:

$$b((u, q), v) \le \underbrace{\left(\sum_{j=1}^{N^{\mathrm{el}}} \|u\|^2_{L(K_j)} + \|q\|^2_{\tilde{W}(\Omega_h)}\right)^{\frac{1}{2}}}_{\|(u,q)\|_{opt}} \times \underbrace{\left(\sum_{j=1}^{N^{\mathrm{el}}} \left\|\tilde{T}v\right\|^2_{L(K_j)} + \|[\![v]\!]\|^2_{\partial\Omega_h}\right)^{\frac{1}{2}}}_{\|v\|_{opt}},$$

where the subscript *opt* denotes the "natural optimal" norms in trial and test spaces correspondingly. Here, we have defined

$$\|[\![v]\!]\|_{\partial\Omega_h} = \sup_{q \in \tilde{W}(\Omega)} \frac{a(q, v)}{\|q\|_{\tilde{W}}} = \sup_{r \in W(\Omega)} \frac{a(r, v)}{\|r\|_W}. \tag{2.14}$$

At this point, one needs to ensure that the optimal norm generates the same topology as that generated by the canonical norm in the broken graph space $W(\Omega_h)$, where the canonical norm (also called as the localizable optimal norm) is defined as

$$\|v\|_{lopt} = \left(\sum_{j=1}^{N^{\mathrm{el}}} \left\|\tilde{T}v\right\|^2_{L(K_j)} + \|v\|^2_{L(\Omega_h)}.\right)^{\frac{1}{2}}$$

Here is the desired result.

THEOREM 2.7. *For all $v \in W(\Omega_h)$, there holds*

$$c_1 \|v\|_{opt} \le \|v\|_{lopt} \le c_2 \|v\|_{opt},$$

i.e., $\|\cdot\|_{opt}$ and $\|\cdot\|_{lopt}$ are equivalent, and hence generating the same topology in $W(\Omega_h)$.

Proof. Owing to the continuity of B, B_{K_j}, and M from (2.3) and (2.5), it is easy to see that $\|[\![v]\!]\|_{\partial\Omega_h} \le C \|v\|_{lopt}$, and hence the optimal test norm is bounded from above by the localizable optimal test norm. To obtain the converse, we adapt the argument proposed in [13] to our abstract framework. We begin by considering the following equation

$$\begin{cases} \text{Given } v \in W(\Omega_h) \subset L(\Omega). \text{ Seek } w \in W(\Omega) \text{ such that} \\ Tw = v \text{ in } L(\Omega), \text{ and} \\ w \in V = \mathcal{N}(B - M), \end{cases} \tag{2.15}$$

By Theorem 2.1, (2.15) is well-posed and the following estimates hold

$$\mu_0 \left\|w\right\|_{L(\Omega)} \le \left\|v\right\|_{L(\Omega)}, \frac{\mu_0}{1+\mu_0} \left\|w\right\|_{W(\Omega)} \le \left\|v\right\|_{L(\Omega)}.$$

As a result, we have

$$\begin{aligned}
\left\|v\right\|_{L(\Omega)}^2 = \left(Tw, v\right)_{L(\Omega)} &= \sum_{j=1}^{N^{\text{el}}} \left(w, \tilde{T}v\right)_{L(K_j)} + a\left(w, v\right) \\
&\le \sum_{j=1}^{N^{\text{el}}} \left\|w\right\|_{L(K_j)} \left\|\tilde{T}v\right\|_{L(K_j)} + \sup_{r \in W(\Omega)} \frac{a\left(r, v\right)}{\left\|r\right\|_W} \left\|w\right\|_{W(\Omega)} \\
&\le \left(\sum_{j=1}^{N^{\text{el}}} \left\|w\right\|_{L(K_j)}^2 + \left\|w\right\|_{W(\Omega)}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N^{\text{el}}} \left\|\tilde{T}v\right\|_{L(K_j)}^2 + \left\|[\![v]\!]\right\|_{\partial\Omega_h}^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{2}} \frac{1+\mu_0}{\mu_0} \left\|v\right\|_{L(\Omega)} \left\|v\right\|_{opt},
\end{aligned}$$

from which it follows that

$$\left\|v\right\|_{L(\Omega)} \le \frac{1}{\sqrt{2}} \frac{1+\mu_0}{\mu_0} \left\|v\right\|_{opt},$$

which in turns yields the bound that we are looking for

$$\left\|v\right\|_{lopt} \le \sqrt{\frac{2\mu_0^2 + \left(1+\mu_0\right)^2}{2\mu_0^2}} \left\|v\right\|_{opt}.$$

□

We are now in the position to discuss the well-posedness of the DPG formulation.

Theorem 2.8 (Well-posedness of the DPG formulation). *The DPG formulation* (2.10) *is well-posed, and the continuity and inf-sup constants are unity in the optimal norms.*

Proof. Since the equality in $b\left(\left(u, q\right), v\right) \le \left\|\left(u, q\right)\right\|_{opt} \left\|v\right\|_{opt}$ is attainable, Theorem 2.6 in [6] shows that the continuity constant M and the inf–sup constant γ are unity. By the Banach-Nečas-Babuška theorem [17] (also known as the generalized Lax-Milgram theorem [4,25]), the remaining task is to prove the following implication

$$\left(b\left(\left(u, q\right), v\right) = 0, \forall \left(u, q\right) \in L\left(\Omega_h\right) \times \tilde{W}\left(\Omega\right)\right) \Rightarrow v = 0.$$

To this end, take $q = u \in V = \mathcal{N}\left(M - B\right) \subset W\left(\Omega\right)$. Then, applying (2.3) element-by-element the expression $b\left(\left(u, q\right), v\right) = 0$ becomes

$$\left(Tu, v\right)_{L(\Omega)} = 0,$$

which yields $v = 0$ in $L\left(\Omega\right)$ since T is isomorphic from V to L as in Theorem 2.1, and this ends the proof. □

On the other hand, if we would like to control the error in approximating u in the $L-$norm, we can adapt the idea in [16] to our abstract framework. In order to achieve this goal, we begin by considering the following adjoint problem:

$$\left\{\begin{array}{l} \text{Given } u \in L(\Omega)\,.\ \text{Seek } v \in W(\Omega) \text{ such that} \\ \tilde{T}v = u \text{ in } L(\Omega)\,, \text{ and} \\ v \in V^* = \mathcal{N}(B + M^*)\,. \end{array}\right. \tag{2.16}$$

Theorem 2.1 then shows that the adjoint problem (2.16) is well-posed, and let us denote its solution as v_u.

PROPOSITION 2.9. *For all* $(u,q) \in L(\Omega) \times \tilde{W}(\Omega)$, *there holds*

$$b\left((u,q), v_u\right) = \|u\|^2_{L(\Omega)}\,. \tag{2.17}$$

Proof. Since $q, v_u \in W(\Omega)$, using (2.3) we have

$$\begin{aligned} \sum_{j=1}^{N^{\text{el}}} \left\langle B_{K_j} q, v_u \right\rangle_{W'(K_j)\times W(K_j)} &= \sum_{j=1}^{N^{\text{el}}} \left(Tq, v_u\right)_{L(K_j)} - \left(q, \tilde{T}v_u\right)_{L(K_j)} \\ &= \left(Tq, v_u\right)_{L(\Omega)} - \left(q, \tilde{T}v_u\right)_{L(\Omega)} = \left\langle Bq, v_u\right\rangle_{W'(\Omega)\times W(\Omega)}\,. \end{aligned}$$

Now the fact that v_u is the solution of (2.16) yields

$$b\left((u,q), v_u\right) = \left(u, \tilde{T}v_u\right)_{L(\Omega)} + \frac{1}{2}\left\langle (M+B)\,q, v\right\rangle_{W'(\Omega)\times W(\Omega)} = \|u\|^2_{L(\Omega)}$$

□

Let us now use subscript *qopt* to denote "quasi-optimal" norms in trial and test spaces[1]. In particular, if we choose either of the following quasi-optimal norms (or any of their combinations) in the test space, i.e.,

$$\|v_u\|_{qopt(\Omega)} = \|v_u\|_{qopt(\Omega_h)} = \left\{\begin{array}{l} \mu_0 \|v_u\|_{L(\Omega_h)} \\ \frac{\mu_0}{\mu_0+1}\|v_u\|_{W(\Omega_h)} \end{array}\right.\,, \tag{2.18}$$

then the corresponding quasi-optimal norm in the trial space reads

$$\|(u,q)\|_{qopt(\Omega_h)} = \sup_{v\in W(\Omega_h)} \frac{b\left((u,q), v\right)}{\|v\|_{qopt(\Omega_h)}}.$$

We can now bound the L-norm of u by the quasi-optimal norm.

THEOREM 2.10. *There holds*

$$\|u\|_{L(\Omega_h)} \le \|(u,q)\|_{qopt(\Omega_h)}\,. \tag{2.19}$$

Proof. From (2.18) and the definitions of quasi-optimal norms we have

$$\begin{aligned} \|u\|^2_{L(\Omega_h)} &= \|v_u\|_{qopt(\Omega)} \frac{b\left((u,q), v_u\right)}{\|v_u\|_{qopt(\Omega_h)}} \\ &\le \|v_u\|_{qopt(\Omega)} \sup_{v\in W(\Omega_h)} \frac{b\left((u,q), v\right)}{\|v\|_{qopt(\Omega_h)}} = \|v_u\|_{qopt(\Omega)} \|(u,q)\|_{qopt(\Omega_h)}\,, \end{aligned}$$

[1] Note that our notion of quasi-optimal norms is not the same as that in [15] since they are not robustly equivalent to the optimal norms

and the stability result in Theorem 2.1 ends the proof. □

It should be pointed out that if we take $\|v\|_{qopt} = \frac{\mu_0}{\mu_0+1}\|v\|_{W(\Omega_h)}$, then it is a scaled version of the localizable optimal norm $\|v\|_{lopt}$. On the other hand, if $\|v\|_{qopt}$ is a function of only $\mu_0\|v\|_{L(\Omega_h)}$, which is weaker than $\|v\|_{lopt}$, the test space $W(\Omega_h)$ is no longer complete in $\|v\|_{qopt}$, and hence should be avoided.

To the rest of the paper, for convenience, we denote the DPG method with optimal norms as DPGopt, with localizable optimal norms as DPGlopt, and with the quasi-optimal norms as DPGqopt.

2.3. Convergence of ideal DPG methods. Let us now denote $\mathcal{U} = L(\Omega_h) \times \tilde{W}(\Omega)$ and $\mathcal{V} = W(\Omega_h)$. Given a set of N independent basis functions $\{\varphi_i\}_{i=1}^N$ in the trial space $\mathcal{U}$, the corresponding optimal test functions $\psi_i = S\varphi_i \in \mathcal{V}, i = 1, \dots, N$, images of the *trial-to-test* operator S [10], can be computed by solving the following equation

$$(\psi_i, v)_{\mathcal{V}} = b(\varphi_i, v), \quad \forall v \in \mathcal{V}, \tag{2.20}$$

where $\|\cdot\|_{\mathcal{V}} \in \left\{\|\cdot\|_{\text{opt}}, \|\cdot\|_{\text{lopt}}, \|\cdot\|_{\text{qopt}}\right\}$ is a norm in $\mathcal{V}$. Here, it is assumed that solving for ψ_i can be done exactly. Since our DPG formulation (2.10) is well-posed as proved in Theorem 2.8, S is bijective and hence $\{\psi_i\}_{i=1}^N$ is also a set of N independent basis functions in $\mathcal{V}$. Let us denote $\mathcal{U}_N = \text{span}\{\varphi_i\}_{i=1}^N$, $\mathcal{V}_N = \text{span}\{\psi_i\}_{i=1}^N$, and (u_N, q_N) be the solution of

$$\left\{\begin{array}{l} \text{Seek } (u_N, q_N) \in \mathcal{U}_N \text{ such that} \\ b((u_N, q_N), v) = \ell(v), \quad \forall v \in \mathcal{V}_N. \end{array}\right. \tag{2.21}$$

Note that the well-posedness of this discrete equation is inherited from the continuous setting (2.10); see [6,10,11] for the detailed exposition. Then the following convergence result is standard.

THEOREM 2.11. *Let $\|\cdot\|_X, \|\cdot\|_Y \in \left\{\|\cdot\|_{opt}, \|\cdot\|_{lopt}, \|\cdot\|_{qopt}\right\}$ be two norms in $\mathcal{V}$ such that*

$$c_1\|v\|_X \le \|v\|_Y \le c_2\|v\|_X, \quad \forall v \in \mathcal{V}.$$

If the test basis functions $\{\psi_i\}_{i=1}^N$ are computed using the $\|\cdot\|_Y$-norm for the test space $\mathcal{V}$, then

$$\|(u,q) - (u_N, q_N)\|_X \le \frac{c_2}{c_1} \inf_{(w,p)\in\mathcal{U}_N} \|(u,q) - (w,p)\|_X.$$

Proof. See [26] for a proof. □

Clearly the error is optimal if we use the $\|\cdot\|_X$-norm for the test space $\mathcal{V}$ to compute the test basis functions. Furthermore, the stiffness matrix of the discrete problem is always symmetric positive definite. Indeed, the symmetry and the positive definiteness are direct consequences of the inner product in $\mathcal{V}$, i.e.,

$$b(\varphi_i, S\varphi_j) = (S\varphi_i, S\varphi_j)_{\mathcal{V}} = (S\varphi_j, S\varphi_i)_{\mathcal{V}} = b(\varphi_j, S\varphi_i).$$

The abstract DPG theory that we have developed is valid for a general class of operators T and $\tilde{T}$ satisfying (2.1a), (2.1b), and the well-posedness in Theorem 2.1. In particular, the abstract DPG formulation (2.10) is well-posed as shown in Theorem 2.8. We shall show that Friedrichs' operators satisfy all the conditions of T and hence the abstract DPG theory developed in this section holds for Friedrichs' systems.

2.4. Comments on the convergence of practical DPG methods. It is assumed in Section 2.3 that we can solve for the test basis functions $\{\psi_i\}_{i=1}^N$ exactly. In practice, we approximate ψ_i by $\tilde{S}\varphi_i$, where $\tilde{S}$ is an approximation of S [23], namely, we replace (2.20) by

$$\left(\tilde{S}\varphi_i, v\right)_{\mathcal{V}} = b\left(\varphi_i, v\right), \quad \forall v \in \mathcal{V}_r \subset \mathcal{V}.$$

As a consequence, the discrete well-posedness is no longer inherited from the continuous one as in the ideal DPG methods. However, following [23], if we further assume that there exists a linear operator $\Pi : \mathcal{V} \to \mathcal{V}_r$ such that, for all $v \in \mathcal{V}$, the following holds

$$b\left(\left(w,p\right), v - \Pi v\right) = 0, \quad \forall \left(w,p\right) \in \mathcal{U}_N,$$
$$\left\|\Pi v\right\|_Y \le c_3 \left\|v\right\|_Y,$$

then we still have discrete well-posedness and convergence.

THEOREM 2.12. *The discrete problem*

$$b\left(\left(u_N, q_N\right), v\right) = \ell\left(v\right), \quad \forall v \in \mathcal{V}_N = span\left\{\tilde{S}\varphi_i\right\}_{i=1}^N$$

is well-posed, and the following convergence result holds

$$\left\|(u,q) - (u_N, q_N)\right\|_X \le c_3 \frac{c_2}{c_1} \inf_{(w,p)\in\mathcal{U}_N} \left\|(u,q) - (w,p)\right\|_X.$$

Proof. See [23] for a proof. □

3. Friedrichs' systems of first order PDEs. Let $\mathcal{D}\left(\Omega\right) = C_0^\infty\left(\Omega\right)$ be the space of test functions. If we set $L = \left[L^2\left(\Omega\right)\right]^m$, $m \in \mathbb{N}$, and $\mathscr{D} = \left[\mathcal{D}\left(\Omega\right)\right]^m$, then $\mathscr{D}$ is dense in L. The following assumptions for Friedrichs' system are standard [18,22]:

$$C \in \left[L^\infty\left(\Omega\right)\right]^{m,m}, \tag{3.1a}$$

$$A^k \in \left[L^\infty\left(\Omega\right)\right]^{m,m}, \quad k = 1,\dots,d, \quad \text{and} \sum_{k=1}^d \partial_k A^k \in \left[L^\infty\left(\Omega\right)\right]^{m,m}, \tag{3.1b}$$

$$A^k = \left(A^k\right)^T \text{ a.e. in } \Omega, \quad k = 1,\dots,d, \tag{3.1c}$$

$$C + C^* - \sum_{k=1}^d \partial_k A^k \ge 2\mu_0 I_m \text{ a.e. in } \Omega, \tag{3.1d}$$

where I_m is the $m \times m$ identity matrix.

Next, we define $T : \mathscr{D} \to L$ as

$$T\varphi = \sum_{k=1}^d A^k \partial_k \varphi + C\varphi, \quad \forall \varphi \in \mathscr{D},$$

and its formal adjoint $\tilde{T} : \mathscr{D} \to L$ as

$$\tilde{T}\varphi = -\sum_{k=1}^d A^k \partial_k \varphi + \left(C^* - \sum_{k=1}^d \partial_k A^k\right)\varphi, \quad \forall \varphi \in \mathscr{D}.$$

Then, it is obvious to see that T and $\tilde{T}$ satisfy (2.1a), (2.1b) and (2.6). Consequently, all the results in Section 2 hold for Friedrichs' systems satisfying (3.1).

For the sake of practicality, it is convenient to study and use the trace of functions in W. Let us assume that

$$\mathcal{B} = \sum_{k=1}^{d} n_k A^k$$

is well-defined a.e. on $\partial\Omega$ with $\mathbf{n} = (n_1, \ldots, n_d)$ being the unit outward normal vector of $\partial\Omega$. For simplicity in writing, let us set $\mathcal{H}^s = [H^s]^m$, where H^s is the usual Sobolev space of order s, and $\mathcal{C}^1 = \left[C^1\right]^m$, where C^1 is the space of continuously differentiable functions. The following representation result for the boundary operator B can be found in [1,24].

Lemma 3.1. *For $u, v \in \mathcal{H}^1(\Omega) \subset W$, there holds*

$$\langle Bu, v\rangle_{W' \times W} = \langle \mathcal{B}u, v\rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega) \times \mathcal{H}^{\frac{1}{2}}(\partial\Omega)}.$$

In particular, for $u, v \in \mathcal{C}_0^\infty\left(\mathbb{R}^d\right)$,

$$\langle Bu, v\rangle_{W' \times W} = \int_{\partial\Omega} v^T \mathcal{B} u \; ds.$$

If Ω has segment property [1], which is true for Lipschitz domains, then $\mathcal{C}^1$ is dense in $\mathcal{H}^1(\Omega)$ which in turn is dense in W, and hence the representation can be uniquely extended the whole space W, i.e.,

$$\langle Bu, v\rangle_{W' \times W} = \langle \mathcal{B}u, v\rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega) \times \mathcal{H}^{\frac{1}{2}}(\partial\Omega)}, \forall u \in W, v \in \mathcal{H}^1(\Omega). \tag{3.2}$$

The definition (2.3) can be therefore considered as the integration by parts formula. It is important to point out that the map $\mathcal{B} : W \rightarrow \mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$ is not surjective in general [1]. Moreover, the range of $\mathcal{B}$ is generally not closed in $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$. Owing to this fact, the boundary operator B may be more preferable since its nullspace W_0 is well-defined and its range space $W_0^\perp$ is obviously closed. It is the key that we explore in this paper. In particular, the construction of the abstract DPG method (2.10) using the boundary operator is twofold. First, we avoid the technicality of specifying the trace of functions in an abstract graph space $W(\Omega)$, allowing the DPG theory to be developed for abstract operators T and $\tilde{T}$. Second, the well-posedness of the resulting general DPG method can be established in a straightforward manner. Nevertheless, when T is specialized for a particular PDE under consideration, the graph space, and hence the trace operator, is often much nicer, as we shall show.

4. Friedrichs' systems of first order PDEs with partial coercivity. In this section we relax the positivity condition (3.1d) to account for Friedrich's systems that have two field structures with partial coercivity. The results in this section are useful for convection-diffusion, Laplace, and linearized continuum mechanics (e.g. linearized compressible elasticity or linearized compressible Navier-Stokes) equations, to name a few. Our development is inspired and based on the well-posedness results of Friedrichs' systems studied in [20].

Assume that there exist two positive integers $m_{\boldsymbol{\sigma}}$ and m_u such that $m = m_{\boldsymbol{\sigma}} + m_u$. Denote $L_{\boldsymbol{\sigma}} = \left[L^2(\Omega)\right]^{m_{\boldsymbol{\sigma}}}$, $L_u = \left[L^2(\Omega)\right]^{m_u}$, and $L = L_{\boldsymbol{\sigma}} \times L_u$. For any $w \in L$, the group variable notion $w = (w^{\boldsymbol{\sigma}}, w^u)$ is used throughout. We decompose C and A accordingly

$$C = \left[\begin{array}{cc} C^{\boldsymbol{\sigma}\boldsymbol{\sigma}} & C^{\boldsymbol{\sigma}u} \\ C^{u\boldsymbol{\sigma}} & C^{uu} \end{array}\right], \quad A^k = \left[\begin{array}{cc} A^{\boldsymbol{\sigma}\boldsymbol{\sigma},k} & E^k \\ \left(E^k\right)^T & G^k \end{array}\right].$$

The following assumptions are important for the well-posedness of our two-field Friedrichs' systems with partial coercivity [20]:

$$\forall k \in 1, \ldots, d, \quad A^{\boldsymbol{\sigma}\boldsymbol{\sigma},k} = 0, \tag{4.1a}$$
$$\exists c_0 > 0, \quad C^{\boldsymbol{\sigma}\boldsymbol{\sigma}} \ge c_0 I_{m_{\boldsymbol{\sigma}}}, \tag{4.1b}$$
$$\left(\left(C + C^* - \sum_{k=1}^d \partial_k A^k\right) z, z\right) \gtrsim \left\|z^{\boldsymbol{\sigma}}\right\|^2_{L_{\boldsymbol{\sigma}}} \text{ a.e. in } \Omega, \tag{4.1c}$$
$$C^{\boldsymbol{\sigma}u} = \left(C^{u\boldsymbol{\sigma}}\right)^* = 0 \text{ and } E^k \text{ are constant over } \Omega, \tag{4.1d}$$
$$\forall z \in V \cup V^*, \left\|z^u\right\|_{L_u} \lesssim \tilde{b}\left(z, z\right)^{\frac{1}{2}} + \left\|E z^u\right\|_{L_{\boldsymbol{\sigma}}} \tag{4.1e}$$

where

$$\tilde{b}\left(u, v\right) = \left(Tu, v\right)_L + \frac{1}{2}\left\langle\left(M - B\right) u, v\right\rangle_{W' \times W},$$

and

$$E = \sum_{k=1}^d E^k \partial_k.$$

Note that the condition (4.1e) is meaningful owing to the positive definiteness of $\tilde{b}$ on W. Here, the notation $z \gtrsim u$ (similarly for $z \lesssim u$) means $z \ge \alpha u$ for some positive constant α.

The following well-posedness result is proved in [20].

Theorem 4.1. *The well-posedness results in Theorem 2.1 still hold if condition* (3.1d) *is replaced by* (4.1).

Since we directly use the well-posedness of equation $Tu = f$ (and its adjoint) to develop the abstract DPG framework instead of the positiveness condition (2.6), all the results for abstract DPG methods in Section 2 hold for Friedrichs' systems with partial coercivity as well. It should be pointed out that, due to condition (4.1), we no longer have explicit expressions for the stability constants as in Theorem 2.1. On the other hand, one can rewrite condition (4.1) with explicit constants, of which it is possible to keep track; but it is not necessary for the purpose of this paper.

5. Examples. For each set of PDEs considered in this section, we first convert the governing equations to first order system (if necessary), followed by a trace theorem (if available), and the detailed specifications of B, M, V and V^*. We then discuss a continuous extension of $(M - B)\, q$ from $W'(\Omega)$ to $W'(\Omega_h)$ along with the compatibility condition (2.11) and the space Q. One of the main results of our analysis is the equivalence of a vector in the quotient space, $q \in W/Q(\Omega)$, and its trace on the skeleton, thus making our DPG formulation practical. Finally, the DPG formulation specialized to the corresponding PDE is presented, followed by a discussion on the relation of our DPG methods and the existing ones in the literature. As will be shown, we recover several existing DPG methods and discover new ones for not only the PDEs that have been already studied but also those that have not been tackled by the DPG community.

We denote the skeleton of the mesh by $\Gamma_h = \cup_{j=1}^{N^{\rm el}} \partial K_j$; the set of all (uniquely defined) faces/edges e, each of which comes with a normal vector $\mathbf{n}_e$. The internal skeleton is then defined as $\Gamma_h^0 = \Gamma_h \setminus \partial\Omega$. If a face/edge $e \in \Gamma_h$ is the intersection of ∂K_i and ∂K_j, $i \neq j$, we define the following jumps:

$$[\![v]\!] = \operatorname{sgn}\left(\mathbf{n}^-\right) v^- + \operatorname{sgn}\left(\mathbf{n}^+\right) v^+, \quad [\![\boldsymbol{\tau}]\!] = \mathbf{n}^- \cdot \boldsymbol{\tau}^- + \mathbf{n}^+ \cdot \boldsymbol{\tau}^+,$$

where

$$\operatorname{sgn}\left(\mathbf{n}^\pm\right) = \left\{ \begin{array}{ll} 1 & \text{if } \mathbf{n}^\pm = \mathbf{n}_e \\ -1 & \text{if } \mathbf{n}^\pm = -\mathbf{n}_e \end{array} \right. .$$

For e belonging to the domain boundary $\partial\Omega$, we define

$$[\![v]\!] = v, \quad [\![\boldsymbol{\tau}]\!] = \mathbf{n}_e \cdot \boldsymbol{\tau}.$$

Note that we allow the arbitrariness in assigning "-" and "+" quantities to the adjacent elements K_i and K_j.

To the rest of the paper, we use the same notation for both a function and its trace (if it is well-defined) when there is no ambiguity.

5.1. Scalar advection-reaction equations. We consider the following scalar hyperbolic PDE over a Lipschitz domain Ω:

$$\begin{aligned} \boldsymbol{\beta} \cdot \nabla u + \mu u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega^-, \end{aligned}$$

where $\partial\Omega^\pm = \{\mathbf{x} \in \partial\Omega : \mp\boldsymbol{\beta}\cdot\mathbf{n} < 0\}$, $\boldsymbol{\beta} \in \left[W^{1,\infty}(\Omega)\right]^d$, $\mu \in L^\infty(\Omega)$, and

$$g \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\partial\Omega^-\right) = \left\{ v : \|v\|^2_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\partial\Omega^-)} = \int_{\partial\Omega^-} |\boldsymbol{\beta}\cdot\mathbf{n}| \, |v|^2 \, ds < \infty \right\}.$$

For convenience in writing, we also define $\Gamma_h^\pm = \Gamma_h \setminus \partial\Omega^\mp$. We assume there exists $\mu_0 > 0$ such that

$$\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq \mu_0 > 0, \quad \text{a.e in } \Omega. \tag{5.1}$$

Note that assumption (5.1) is not a limitation since it is always valid under a change of variable with exponential factor [22, 24]. Clearly, the graph space is given by

$$W(\Omega) = \left\{ u \in L^2(\Omega) : \boldsymbol{\beta}\cdot\nabla u \in L^2(\Omega) \right\} = H^1_{\boldsymbol{\beta}}(\Omega).$$

This is a particular instance of Friedrichs' systems considered in Section 3 with $m = 1$, $C = \mu$ and $A^k = \beta_k$, where β_k is the kth component of vector $\boldsymbol{\beta}$. The following proposition summarizes some of the results in [18, 21].

LEMMA 5.1. *Assume that $\partial\Omega^-$ and $\partial\Omega^+$ is well-separated, i.e., dist$(\partial\Omega^-, \partial\Omega^+) > 0$. Then the following hold:*

i) The trace operator $\gamma : H^1_{\boldsymbol{\beta}}(\Omega) \to L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\partial\Omega)$ is a continuous surjection.

ii) $\mathcal{B} = \boldsymbol{\beta}\cdot\mathbf{n}$ and the boundary operator B satisfies

$$\langle Bu, v\rangle_{W'(\Omega)\times W(\Omega)} = \int_{\partial\Omega} \boldsymbol{\beta}\cdot\mathbf{n} uv \, ds, \quad \forall u, v \in H^1_{\boldsymbol{\beta}}(\Omega).$$

iii) Define $\langle Mu, v\rangle_{W'(\Omega)\times W(\Omega)} = \int_{\partial\Omega} |\boldsymbol{\beta}\cdot\mathbf{n}|\, uv\, ds$, *then* M *satisfies* (2.5a) *and* (2.5b). *Furthermore,*

$$V = \left\{v \in H^1_{\boldsymbol{\beta}}(\Omega) : \boldsymbol{\beta}\cdot\mathbf{n}\,(v-g) = 0 \text{ on } \partial\Omega^-\right\} \text{ and}$$
$$V^* = \left\{v \in H^1_{\boldsymbol{\beta}}(\Omega) : \boldsymbol{\beta}\cdot\mathbf{n}v = 0 \text{ on } \partial\Omega^+\right\}.$$

What remains to be studied are the compatibility condition and the quotient space $\tilde{H}^1_{\boldsymbol{\beta}}(\Omega) = H^1_{\boldsymbol{\beta}}/Q(\Omega)$. We assume that the mesh satisfies the separation condition in Lemma 5.1, namely, ∂K_j^- and ∂K_j^+ is well-separated for all $j = 1,\dots,N^{\text{el}}$ (note that this is only a sufficient condition). Without loss of generality, it is assumed that $\boldsymbol{\beta}\cdot\mathbf{n} \neq 0$ a.e. on Γ_h^+ in the following theorem since otherwise we can always redefine Γ_h^+ by taking away any non-trivial measure subsets of Γ_h+ on which $\boldsymbol{\beta}\cdot\mathbf{n} = 0$. Using results of B, M and the trace operator in Lemma 5.1, a natural extension of $(M-B)\,q$ from $W'(\Omega)$ to $W'(\Omega_h)$ is specified as

$$\langle (M-B)\,q, v\rangle_{\left[H^1_{\boldsymbol{\beta}}(\Omega_h)\right]'\times H^1_{\boldsymbol{\beta}}(\Omega_h)} = -2\sum_{e\in\partial\Omega_h^-}\int_e \boldsymbol{\beta}\cdot\mathbf{n}qv\; ds,$$

for any $q \in H^1_{\boldsymbol{\beta}}(\Omega)$ and $v \in H^1_{\boldsymbol{\beta}}(\Omega_h)$. Consequently, the compatibility condition (2.11) is trivial. We next study the quotient space $\tilde{H}^1_{\boldsymbol{\beta}}(\Omega) = H^1_{\boldsymbol{\beta}}/Q(\Omega)$.

Theorem 5.2.

i) $Q = \left\{q \in H^1_{\boldsymbol{\beta}}(\Omega) : q = 0 \text{ on } \Gamma_h^+\right\}$. *Furthermore,* $H^1_{\boldsymbol{\beta}}/Q(\Omega)$ *is isomorphic to* $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$. *In particular, the trace of a function in the quotient space* $H^1_{\boldsymbol{\beta}}/Q(\Omega)$ *is independent of its representations.*

ii) For each $\hat{u} \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$, *define a new norm*

$$\|\hat{u}\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)} = \|[q]\|_{H^1/Q(\Omega)},$$

where $[q] \in H^1/Q(\Omega)$ *such that there exists a representation* q *satisfying* $q = \hat{u}$ *on* Γ_h^+. *Then,* $\|\cdot\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}$ *is equivalent to* $\|\cdot\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}$, *and hence generating the same topology in* $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$. *In particular,* $H^1_{\boldsymbol{\beta}}/Q(\Omega)$ *is homeomorphic to* $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$.

Proof.

i) The results in Lemma 5.1 allow us to write $a(q,v)$ as

$$a(q,v) = \int_{\Gamma_h^+} |\boldsymbol{\beta}\cdot\mathbf{n}|\, q[\![v]\!]\; ds = \sum_{e\in\Gamma_h^+}\int_e |\boldsymbol{\beta}\cdot\mathbf{n}|\, q[\![v]\!]\; ds,$$

and to conclude that $\gamma : H^1_{\boldsymbol{\beta}}/Q(\Omega) \to L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$ is surjective. Clearly, $a(q,v) = 0, \forall v \in H^1_{\boldsymbol{\beta}}(\Omega_h)$ implies that $\gamma q = 0$ on any subset of Γ_h^+, and hence the first assertion follows. The injectivity of γ can be shown as follows. Let $q_1, q_2 \in H^1_{\boldsymbol{\beta}}/Q(\Omega)$ such that their traces on Γ_h^+ are the same. Then one has

$$a(q_1-q_2, v) = \sum_{e\in\Gamma_h^+}\int_e |\boldsymbol{\beta}\cdot\mathbf{n}|\,(q_1-q_2)\,[\![v]\!]\; ds = 0, \quad \forall v \in H^1_{\boldsymbol{\beta}}(\Omega_h),$$

which implies $q_1 = q_2$ in $H^1_{\boldsymbol{\beta}}/Q(\Omega)$.

ii) The definition of the new norm is meaningful due to $i)$ and the definition of norm in the quotient space. Now, since $\gamma : q \mapsto \gamma q$ is a continuous surjection from $H^1_{\boldsymbol{\beta}}/Q(\Omega)$ to $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$, we have

$$\|\gamma q\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)} \le c_2 \|q\|_{H^1_{\boldsymbol{\beta}}/Q(\Omega)} = c_2 [\![\gamma q]\!]_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)}.$$

On the other hand, since $H^1_{\boldsymbol{\beta}}/Q(\Omega)$ and $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$ are Banach spaces, and γ is bijective, a direct consequence of the Open Mapping theorem [25] shows that

$$\|\gamma q\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)} \ge c_1 \|q\|_{H^1_{\boldsymbol{\beta}}/Q(\Omega)} = c_1 [\![\gamma q]\!]_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)}.$$

Thus, the equivalence of the norms and the homeomorphism follow.

□

As a direct consequence of Theorem 5.2, we can identify $q \in H^1_{\boldsymbol{\beta}}/Q(\Omega)$ with $\hat{u} \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$, and we can use either $[\![\cdot]\!]_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)}$ or $\|\cdot\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)}$ as norm in $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$. The ultra weak formulation (2.10) can be now written equivalently as:

Given $g \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\partial\Omega^-\right), f \in \left[H^1_{\boldsymbol{\beta}}(\Omega_h)\right]'$. Seek $(u, \hat{u}) \in L(\Omega_h) \times L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\Gamma_h^+\right)$ such that

$$\begin{aligned}
&\sum_{j=1}^{N^{\rm el}} \int_{K_j} u\left(-\nabla\cdot(\boldsymbol{\beta} v) + \mu v\right)\, d\mathbf{x} + \sum_{e\in\Gamma_h^+} \int_e |\boldsymbol{\beta}\cdot\mathbf{n}|\, \hat{u} [\![v]\!]\, ds \\
&= \langle f, v\rangle_{\left[H^1_{\boldsymbol{\beta}}(\Omega_h)\right]'\times H^1_{\boldsymbol{\beta}}(\Omega_h)} - \int_{e\in\partial\Omega^-} \boldsymbol{\beta}\cdot\mathbf{n} g v\, ds, \quad \forall v \in H^1_{\boldsymbol{\beta}}(\Omega_h). \qquad (5.2)
\end{aligned}$$

It follows that all the results in Section 2 hold for (5.2). In particular, the DPGopt, DPGlopt, and DPGqopt methods are well-posed; the DPGopt coincides with the second DPG method analyzed in [6], the DPGlopt recovers the DPG method used, but not analyzed, in [11] for two dimensional transport equation, and the DPGqopt has not been studied in the literature. The beauty of the abstract formulation here is that the well-posedness of all three DPG methods is immediately available for transport equations in any dimensions.

5.2. Convection-diffusion-reaction equations. The problem of interest in this section is the following:

$$\begin{aligned}
-\nabla\cdot(\varepsilon\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where we assume $\boldsymbol{\beta} \in \left[L^\infty(\Omega)\right]^d$, $\nabla\cdot\boldsymbol{\beta} \in L^\infty(\Omega)$, and ε is $d \times d$ symmetric positive definite matrix with smallest eigenvalue uniformly bounded away from zero. We first rewrite the equation in the first order form as

$$\varepsilon^{-1}\boldsymbol{\sigma} + \nabla u = 0 \quad \text{in } \Omega, \qquad (5.3a)$$

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{\beta}\cdot\nabla u + \mu u = f \quad \text{in } \Omega, \qquad (5.3b)$$

$$u = 0 \quad \text{on } \partial\Omega. \qquad (5.3c)$$

We now relax condition (5.1) by the following weaker assumption

$$\operatorname*{ess\,inf}_{\Omega}\left(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\right) \ge 0,$$

then it is trivial to see that condition (4.1c) holds. What remains to be check is condition (4.1e), but this is immediate by the Poincaré inequality. Consequently, (5.3) is a particular instance of Friedrichs' system with partial coercivity introduced in Section 4 with $m = d + 1$ and the corresponding matrices:

$$C = \begin{bmatrix} \varepsilon^{-1} & \mathbf{0} \\ \mathbf{0} & \mu \end{bmatrix}, \quad A^k = \begin{bmatrix} \mathbf{0} & \mathbf{e}^k \\ \left(\mathbf{e}^k\right)^T & \beta^k \end{bmatrix},$$

where e^k the kth column of the $d \times d$ identity matrix, and $\mathbf{0}$ zero matrices with appropriate size. It is not difficult to see that the graph space is given by

$$W = H\left(\text{div}, \Omega\right) \times H^1\left(\Omega\right)$$

due to the following norm equivalence

$$\begin{aligned} c_1 \left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{\sigma}\|_{L^2(\Omega)} \right) &\le \|\nabla u\|_{L^2(\Omega)} + \|\boldsymbol{\beta} \cdot \nabla u + \nabla \cdot \boldsymbol{\sigma}\|_{L^2(\Omega)} \\ &\le c_2 \left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{\sigma}\|_{L^2(\Omega)} \right). \end{aligned}$$

The following proposition summarizes some of the results in [18].

Lemma 5.3.

i) The trace operator

$$\gamma : H\left(div, \Omega\right) \times H^1\left(\Omega\right) \ni \left(\boldsymbol{\sigma}, u\right) \mapsto \left(\boldsymbol{\sigma} \cdot \mathbf{n}, u\right) \in H^{-\frac{1}{2}}\left(\partial\Omega\right) \times H^{\frac{1}{2}}\left(\partial\Omega\right)$$

is a continuous surjection satisfying

$$\begin{aligned} \left\langle B\left(\boldsymbol{\sigma}, u\right), \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = &\left\langle \boldsymbol{\sigma} \cdot \mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + \\ &\left\langle \boldsymbol{\tau} \cdot \mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + \int_{\partial\Omega} \boldsymbol{\beta} \cdot \mathbf{n} uv \; ds. \end{aligned}$$

ii) Define

$$\begin{aligned} \left\langle M\left(\boldsymbol{\sigma}, u\right), \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = &\left\langle \boldsymbol{\sigma} \cdot \mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} - \\ &\left\langle \boldsymbol{\tau} \cdot \mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}, \end{aligned}$$

then M satisfies (2.5a) and (2.5b). Furthermore,

$$V = V^* = \left\{ \left(\boldsymbol{\sigma}, u\right) \in H\left(div, \Omega\right) \times H^1\left(\Omega\right) : u|_{\partial\Omega} = 0 \right\} = H\left(div, \Omega\right) \times H^1_0\left(\Omega\right).$$

For any $q = \left(q^{\boldsymbol{\sigma}}, q^u\right) \in W\left(\Omega\right)$, Lemma 5.6 suggests that a natural extension of $\left(M - B\right) q$ from $W'\left(\Omega\right)$ to $W'\left(\Omega_h\right)$ be given by

$$\begin{aligned} \left\langle \left(M - B\right) q, \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} = &-2\left\langle \boldsymbol{\tau} \cdot \mathbf{n}, q^u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)} \\ &- \int_{\partial\Omega_h} \boldsymbol{\beta} \cdot \mathbf{n} q^u v \; ds, \end{aligned}$$

from which the compatibility condition (2.11) is trivial.

As shown in [18], the boundary matrix M is not unique. In fact there are infinite of them, and our choice is probably the simplest. Next, we study the quotient space $\tilde{W}\left(\Omega\right) = H\left(\text{div}\right) \times H^1/Q\left(\Omega\right)$. As in Section 5.1, we assume that if $\boldsymbol{\beta}$ is not identically zero then $\boldsymbol{\beta} \cdot \mathbf{n} \neq 0$ a.e. on Γ_h. Here is a result parallel to Theorem 5.2.

Theorem 5.4.

i) The subspace Q is given by

$$Q = \left\{ q \in H\left(div, \Omega\right) \times H^1\left(\Omega\right) : \left(q^{\boldsymbol{\sigma}} \cdot \mathbf{n}, q^u\right) = 0 \text{ on } \Gamma_h^0 \text{ and } \right.$$
$$\left. q^{\boldsymbol{\sigma}} \cdot \mathbf{n} = -\frac{1}{2}\left|\boldsymbol{\beta}\cdot\mathbf{n}\right| q^u \text{ in } H^{-\frac{1}{2}}\left(\partial\Omega_h\right) \right\}.$$

Furthermore, $H\left(div\right) \times H^1/Q\left(\Omega\right)$ is isomorphic to $H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$. In particular, the trace of a function in the quotient space $H\left(div\right) \times H^1/Q\left(\Omega\right)$ is independent of its representations.

ii) For each $\left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \in H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$, define a new norm

$$\left\| \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \right\|_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)} = \left\|[q]\right\|_{H(div)\times H^1/Q(\Omega)},$$

where $[q] \in H\left(div\right) \times H^1/Q\left(\Omega\right)$ such that there exists a presentation q of $[q]$ satisfying $\gamma q = \left(q^{\boldsymbol{\sigma}} \cdot \mathbf{n}, q^u\right) = \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right)$ on Γ_h. Then, $\left\| \cdot \right\|_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)}$ is equivalent to $\left\|\cdot\right\|_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)}$, and hence generating the same topology in $H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$. In particular, $H\left(div\right) \times H^1/Q\left(\Omega\right)$ and $H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$ are homeomorphic.

Proof. For this example, one has

$$a\left(q, \left(\boldsymbol{\tau}, v\right)\right) = \sum_{j=1}^{N^{\mathrm{el}}} \frac{1}{2}\int_{\partial K_j} \left|\boldsymbol{\beta}\cdot\mathbf{n}\right| q^u [\![v]\!] \, ds + \left\langle [\![\boldsymbol{\tau}]\!], q^u \right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)}$$
$$+ \sum_{e\in\Gamma_h} \left\langle q^{\boldsymbol{\sigma}} \cdot \mathbf{n}_e, [\![v]\!] \right\rangle_{H^{-\frac{1}{2}}(e)\times H^{\frac{1}{2}}(e)}.$$

The surjectivity of the trace operator allows us to easily show that $a\left(q, \left(\boldsymbol{\tau}, v\right)\right) = 0, \forall \left(\boldsymbol{\tau}, v\right) \in H\left(\mathrm{div}, \Omega_h\right) \times H^1\left(\Omega_h\right)$ implies $q = \left(q^{\boldsymbol{\sigma}} \cdot \mathbf{n}, q^u\right) = 0$ on Γ_h^0 and $q^{\boldsymbol{\sigma}} \cdot \mathbf{n}, = -\frac{1}{2}\boldsymbol{\beta}\cdot\mathbf{n} q^u$ on $H^{-\frac{1}{2}}\left(\Omega_h\right)$. Indeed, take $v = 0$, then it can be deduced that $q^u = 0$ on Γ_h^0. Next, take $v \in H_0^1\left(\Omega_h\right)$, we infer that $q^{\boldsymbol{\sigma}} = 0$ on Γ_h^0, and the second assertion now follows. The rest of the proof is similar to that of Theorem 5.2. ◻

The results are somewhat simpler if there is no convection, i.e., $\boldsymbol{\beta} = \mathbf{0}$. Clearly, this case includes the Poisson equation.

Corollary 5.5. *Assume $\boldsymbol{\beta} = \mathbf{0}$, then:*

i) The subspace Q is given by

$$Q = \left\{ q \in H\left(div, \Omega\right) \times H^1\left(\Omega\right) : q^{\boldsymbol{\sigma}} \cdot \mathbf{n} = 0 \text{ on } \Gamma_h \text{ and } q^u = 0 \text{ on } \Gamma_h^0 \right\}.$$

Furthermore, $H\left(div\right) \times H^1/Q\left(\Omega\right)$ is isomorphic to $H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0\right)$. In particular, the trace of a function in the quotient space $H\left(div\right) \times H^1/Q\left(\Omega\right)$ is independent of its representations.

ii) For each $\left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \in H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0\right)$, define a new norm

$$\left\| \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \right\|_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)} = \left\|[q]\right\|_{H(div)\times H^1/Q(\Omega)},$$

where $[q] \in H\left(div\right) \times H^1/Q\left(\Omega\right)$ such that there exists a representation q satisfying $\left(q^{\boldsymbol{\sigma}} \cdot \mathbf{n}, q^u\right) = \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right)$ on $\Gamma_h \times \Gamma_h^0$. Then, $\left\| \cdot \right\|_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)}$ is equivalent to $\left\|\cdot\right\|_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)}$, and hence generating the same topology in $H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0\right)$. In particular, $H\left(div\right) \times H^1/Q\left(\Omega\right)$ and $H^{-\frac{1}{2}}\left(\Gamma_h\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0\right)$ are homeomorphic.

Proof. It is a direct consequence of Theorem 5.4. ☐

As a direct consequence of Theorem 5.4, we can identify $q \in H(\mathrm{div}) \times H^1/Q(\Omega)$ with $(\hat{\boldsymbol{\sigma}}, \hat{u}) \in H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h)$, and we can use either $\| \cdot \|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$ or $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$ as norm in $H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h)$. The abstract DPG formulation (2.10) now equivalently becomes:

$$
\begin{aligned}
&\text{Given } f \in \left[H^1(\Omega_h)\right]'.\\
&\text{Seek } (\boldsymbol{\sigma}, u, \hat{\boldsymbol{\sigma}}, \hat{u}) \in \left[L^2(\Omega_h)\right]^d \times L^2(\Omega_h) \times H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h) \text{ such that}\\
&\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} \boldsymbol{\sigma}\cdot\left(\varepsilon^{-1}\boldsymbol{\tau} - \nabla v\right) + u\left(-\nabla\cdot\boldsymbol{\tau} - \nabla\cdot(\boldsymbol{\beta} v) + \mu v\right)\, d\mathbf{x} + \frac{1}{2}\int_{\partial K_j} |\boldsymbol{\beta}\cdot\mathbf{n}|\, \hat{u}[\![v]\!]\, ds\\
&+ \langle [\![\boldsymbol{\tau}]\!], \hat{u}\rangle_{H^{-\frac{1}{2}}(\Gamma_h^0)\times H^{\frac{1}{2}}(\Gamma_h^0)} + \langle \hat{\boldsymbol{\sigma}}, [\![v]\!]\rangle_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}\\
&= \langle f, v\rangle_{[H^1(\Omega_h)]'\times[H^1(\Omega_h)]}, \quad \forall (\boldsymbol{\tau}, v) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h).
\end{aligned} \tag{5.4}
$$

On the other hand, if $\boldsymbol{\beta} = \mathbf{0}$, we can identify $q \in H(\mathrm{div})\times H^1/Q(\Omega)$ with $(\hat{\boldsymbol{\sigma}}, \hat{u}) \in H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h^0)$, and we can use either $\| \cdot \|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h^0)}$ or $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h^0)}$ as norm in $H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h^0)$. In this case, the abstract DPG formulation (2.10) now equivalently becomes:

$$
\begin{aligned}
&\text{Given } f \in \left[H^1(\Omega_h)\right]'.\\
&\text{Seek } (\boldsymbol{\sigma}, u, \hat{\boldsymbol{\sigma}}, \hat{u}) \in \left[L^2(\Omega_h)\right]^d \times L^2(\Omega_h) \times H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h^0) \text{ such that}\\
&\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} \boldsymbol{\sigma}\cdot\left(\varepsilon^{-1}\boldsymbol{\tau} - \nabla v\right) + u\left(-\nabla\cdot\boldsymbol{\tau} + \mu v\right)\, d\mathbf{x}\\
&+ \langle [\![\boldsymbol{\tau}]\!], \hat{u}\rangle_{H^{-\frac{1}{2}}(\Gamma_h^0)\times H^{\frac{1}{2}}(\Gamma_h^0)} + \langle \hat{\boldsymbol{\sigma}}, [\![v]\!]\rangle_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}\\
&= \langle f, v\rangle_{[H^1(\Omega_h)]'\times[H^1(\Omega_h)]}, \quad \forall (\boldsymbol{\tau}, v) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h).
\end{aligned} \tag{5.5}
$$

Consequently, results in Section 2 are valid for (5.5). More specifically, the well-posedness of DPGopt, DPGlopt, and DPGqopt is readily available for both (5.4) and (5.5). It should be pointed out that the DPGopt and DPGlopt for (5.5) are identical to those analyzed in [10] for the Poisson equation ($\boldsymbol{\beta} = \mathbf{0}$ and $\mu = 0$) if $f \in L^2(\Omega)$. Here, our approach is novel in the sense that the function spaces and the well-posedness of the corresponding DPG formulation are the direct consequences of the single abstract framework developed in Section 2 for all $f \in \left[H^1(\Omega_h)\right]' \supset L^2(\Omega)$. Again, a single proof of well-posedness is applicable for all PDEs of Friedrichs' type instead of a different and special proof for each PDE, as done in the existing literature. However, we admit the fact that taking advantage of particular structure of a PDE under consideration may yield sharper stability estimates and much more. This is not possible for our abstract and unified framework in Section 2.

It turns out that the DPGlopt coincides with the DPG method used in [11] for convection-diffusion problem ($\mu = 0$) in two dimensions. (Actually, there is a slight difference in imposing the boundary condition for the convection term, i.e, the third term on the right side of (5.4); we have a factor 1/2 at the domain boundary $\partial\Omega_h$ instead of 1 as in [11].) However, while the DPGlopt method is assumed to be well-posed in [11], our results in Section 2 show that it is indeed the case and the proof is the direct consequence of Theorem 2.8. Moreover, our function space setting for $\hat{u}$ comes out naturally from the abstract setting while it is left unspecified in [11]. Recently, the authors of [11] have analyzed their DPG method for convection-diffusion problem in [10] where they combine the diffusion flux $\hat{\boldsymbol{\sigma}}$ and convection flux $|\boldsymbol{\beta}\cdot\mathbf{n}|\,\hat{u}$ into a single unknown total flux. Nevertheless, there is certainly no reason to prevent us from combining diffusion and convection fluxes in (5.4) so that our DPG methods recover those in [10]. Our abstract framework is not able to recover the robust versions of the DPG method developed in [16].

5.3. Linear(ized) continuum mechanics. The problem of interest in this section is governed by

$$\begin{array}{ll} \mathcal{A}\boldsymbol{\sigma} - \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) = 0 & \text{in } \Omega, \\ -\frac{1}{2}\nabla\cdot\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^T\right) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{5.6}$$

where $\mathcal{A}$ is the compliance tensor, u the displacement in solid mechanics or velocity in fluid mechanics, and $\boldsymbol{\sigma}$ the stress tensor. Note that the stress tensor $\boldsymbol{\sigma}$ with values in $\mathbb{R}^{d,d}$ can be identified with a vector-valued field in $\mathbb{R}^{d^2}$. However, to simplify the notations we use the same symbol $\boldsymbol{\sigma}$ for both tensor-valued and vector-valued fields, and this should be clear in each context. Similarly, we identify the tensor $\mathcal{A}$ with a matrix in $\mathbb{R}^{d^2,d^2}$.

Assume that $\mathcal{A}$ is self-adjoint and uniformly positive definite on $\mathbb{R}^{d,d}$ with each component in $L^\infty(\Omega)$. We further assume that

$$\mu_0 = \operatorname*{ess\,inf}_{\Omega}\left(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\right) \geq 0.$$

Set $m = d^2 + d$, $m_{\boldsymbol{\sigma}} = d^2$, and $m_u = d$. Thus, the full coercivity (5.1) does not hold, but the partial coercivity (4.1c) does. It is straightforward to cast (5.6) into the framework of two-field Friedrichs' system in Section 4 with the corresponding matrices:

$$C = \begin{bmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mu I_d \end{bmatrix}, \quad A^k = \begin{bmatrix} \mathbf{0} & \mathcal{E}^k \\ \left(\mathcal{E}^k\right)^T & \beta^k I_d \end{bmatrix},$$

where $\mathcal{E}^k$ is a $\mathbb{R}^{d^2,d}$ matrix defined as $\mathcal{E}^k_{[ij],l} = -\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ where $1 \leq i,j,k \leq d$, and δ is the usual Kronecker symbol. With these specifications, (5.6) satisfies hypotheses (3.1a), (3.1b), and (3.1c) if $\boldsymbol{\beta} \in \left[W^{1,\infty}(\Omega)\right]^d$ and $\mu \in L^\infty(\Omega)$. In general, (3.1d) does not hold unless $\mu_0 > 0$. Fortunately, (4.1c) holds since $C^{\boldsymbol{\sigma}\boldsymbol{\sigma}} = \mathcal{A}$ is uniformly positive definite. What remains to be checked is the assumption (4.1e), but this is clear by the Korn's first inequality. Thus, (5.6) fulfills all the conditions of the two-field Friedrichs' system discussed in Section 4.

Let us denote $H\left(div, \Omega; \mathbb{R}^{d,d}\right) = \left\{\boldsymbol{\sigma} \in L^2\left(\Omega;\mathbb{R}^{d,d}\right) : \nabla\cdot\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^T\right) \in L^2\left(\Omega;\mathbb{R}^d\right)\right\}$, where the divergence operator acts row-wise. Then, the graph space [19, 20] is given by

$$W(\Omega) = H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right).$$

Next, we extract from [20] the properties of B, M, V, V^*, and the trace operator γ.

LEMMA 5.6. *The following hold:*

i) The trace operator γ defined by

$$\gamma : H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega; \mathbb{R}^d\right) \to H^{-\frac{1}{2}}\left(\partial\Omega; \mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\partial\Omega; \mathbb{R}^d\right)$$
$$(\boldsymbol{\sigma}, u) \stackrel{\gamma}{\mapsto} (\boldsymbol{\sigma}\cdot\mathbf{n}, u)$$

is a continuous surjection satisfying

$$\left\langle B\left(\boldsymbol{\sigma}, u\right), \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = -\left\langle \frac{1}{2}\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^T\right)\cdot\mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)}$$
$$-\left\langle \frac{1}{2}\left(\boldsymbol{\tau}+\boldsymbol{\tau}^T\right)\cdot\mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)} + \int_{\partial\Omega} \boldsymbol{\beta}\cdot\mathbf{n} uv\ ds.$$

ii) Define

$$\left\langle M\left(\boldsymbol{\sigma}, u\right), \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = -\left\langle \frac{1}{2}\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^T\right)\cdot\mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)}$$
$$+\left\langle \frac{1}{2}\left(\boldsymbol{\tau}+\boldsymbol{\tau}^T\right)\cdot\mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)}$$

then M satisfies (2.5a) and (2.5b). Furthermore,

$$V = V^* = H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H_0^1\left(\Omega; \mathbb{R}^d\right).$$

For any $q = (q^{\boldsymbol{\sigma}}, q^u) \in W(\Omega)$, Lemma 5.6 suggests that a natural extension of $(M - B)\, q$ from $W'(\Omega)$ to $W'(\Omega_h)$ be given by

$$\left\langle (M-B)\, q, (\boldsymbol{\tau}, v)\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} = \left\langle \left(\boldsymbol{\tau}+\boldsymbol{\tau}^T\right)\cdot\mathbf{n}, q^u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)}$$
$$-\int_{\partial\Omega_h} \boldsymbol{\beta}\cdot\mathbf{n} q^u v\ ds,$$

from which the compatibility condition (2.11) is trivial.

Next, we study the quotient space $\tilde{W}(\Omega) = H\left(\mathrm{div}, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega; \mathbb{R}^d\right)/Q(\Omega)$. As in Section 5.1, we assume that if $\boldsymbol{\beta}$ is not identically zero then $\boldsymbol{\beta}\cdot\mathbf{n} \neq 0$ a.e. on Γ_h. Here is a result parallel to Theorem 5.2.

Theorem 5.7.

i) The subspace Q is given by

$$Q = \left\{ q \in H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega; \mathbb{R}^d\right) : \left(\frac{1}{2}\left(q^{\boldsymbol{\sigma}} + \left(q^{\boldsymbol{\sigma}}\right)^T\right)\cdot\mathbf{n}, q^u\right) = 0 \text{ on } \Gamma_h^0\right.$$
$$\left.\text{and } \left(q^{\boldsymbol{\sigma}} + \left(q^{\boldsymbol{\sigma}}\right)^T\right)\cdot\mathbf{n} = \left|\boldsymbol{\beta}\cdot\mathbf{n}\right| q^u \text{ in } H^{-\frac{1}{2}}\left(\partial\Omega_h; \mathbb{R}^d\right)\right\}.$$

Furthermore, $H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega; \mathbb{R}^d\right)/Q(\Omega)$ is isomorphic to $H^{-\frac{1}{2}}\left(\Gamma_h; \mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h; \mathbb{R}^d\right)$. In particular, the trace of a function in the quotient space $H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega; \mathbb{R}^d\right)/Q(\Omega)$ is independent of its representations.

ii) For each $(\hat{\boldsymbol{\sigma}}, \hat{u}) \in H^{-\frac{1}{2}}\left(\Gamma_h; \mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h; \mathbb{R}^d\right)$, define a new norm

$$\left\| (\hat{\boldsymbol{\sigma}}, \hat{u}) \right\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)} = \left\| [q] \right\|_{H(div,\Omega;\mathbb{R}^{d,d})\times H^1(\Omega;\mathbb{R}^d)/Q(\Omega)},$$

where $[q] \in H\left(div,\Omega;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ *such that there exists a representation* q *satisfying* $\gamma q = \left(\frac{1}{2}\left(q^{\boldsymbol{\sigma}} + \left(q^{\boldsymbol{\sigma}}\right)^T\right)\cdot \mathbf{n}, q^u\right) = \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right)$ *on* Γ_h. *Then,* $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)}$ *is equivalent to* $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)}$, *and hence generating the same topology in* $H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)$. *In particular,* $H\left(div,\Omega;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ *and* $H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)$ *are homeomorphic.*

Proof. For this example, one has

$$a\left(q,\left(\boldsymbol{\tau},v\right)\right) = \sum_{j=1}^{N^{\mathrm{el}}} \frac{1}{2}\int_{\partial K_j} \left|\boldsymbol{\beta}\cdot\mathbf{n}\right| q^u [\![v]\!]\, ds - \left\langle \frac{1}{2}[\![\boldsymbol{\tau} + \boldsymbol{\tau}^T]\!], q^u \right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right);\mathbb{R}^d}$$
$$- \sum_{e\in\Gamma_h} \left\langle \frac{1}{2}\left(q^{\boldsymbol{\sigma}} + \left(q^{\boldsymbol{\sigma}}\right)^T\right)\cdot\mathbf{n}_e, [\![v]\!] \right\rangle_{H^{-\frac{1}{2}}(e;\mathbb{R}^d)\times H^{\frac{1}{2}}(e;\mathbb{R}^d)}.$$

The rest of the proof is similar to that of Theorem 5.4. ☐

The results are somewhat simpler if there is no convection, i.e., $\boldsymbol{\beta} = \mathbf{0}$. Clearly, this case includes the linear elasticity equation.

Corollary 5.8. *Assume* $\boldsymbol{\beta} = \mathbf{0}$, *then:*

i) The subspace Q *is given by*

$$Q = \left\{q \in H\left(div,\Omega;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right) : \left(\frac{1}{2}\left(q^{\boldsymbol{\sigma}} + q^{\boldsymbol{\sigma}}\right)^T\right)\cdot\mathbf{n} = 0 \text{ on } \Gamma_h \right.$$
$$\left. q^u = 0 \text{ on } \Gamma_h^0\right\}.$$

Furthermore, $H\left(div,\Omega;\mathbb{R}^{d,d}\right)\times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ *is isomorphic to* $H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)$. *In particular, the trace of a function in the quotient space* $H\left(div,\Omega;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ *is independent of its representations.*

ii) For each $\left(\hat{\boldsymbol{\sigma}},\hat{u}\right) \in H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)$, *define a new norm*

$$\|\left(\hat{\boldsymbol{\sigma}},\hat{u}\right)\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)} = \left\|[q]\right\|_{H(div,\Omega;\mathbb{R}^{d,d})\times H^1(\Omega;\mathbb{R}^d)/Q(\Omega)},$$

where $[q] \in H\left(div,\Omega;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ *such that there exists a representation* q *satisfying* $\gamma q = \left(\frac{1}{2}\left(q^{\boldsymbol{\sigma}} + \left(q^{\boldsymbol{\sigma}}\right)^T\right)\cdot\mathbf{n}, q^u\right) = \left(\hat{\boldsymbol{\sigma}},\hat{u}\right)$ *on* $\Gamma_h \times \Gamma_h^0$. *Then,* $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}$ *is equivalent to* $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}$, *and hence generating the same topology in* $H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)$. *In particular,* $H\left(div,\Omega;\mathbb{R}^{d,d}\right)\times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ *and* $H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)$ *are homeomorphic.*

Proof. It is a direct consequence of Theorem 5.7. ☐

Theorem 5.7 suggests that we can identify $q \in H\left(\mathrm{div},\Omega;\mathbb{R}^{d,d}\right)\times H^1\left(\Omega;\mathbb{R}^d\right)/Q\left(\Omega\right)$ with $\left(\hat{\boldsymbol{\sigma}},\hat{u}\right) \in H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)\times H^{\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)$. The abstract DPG formulation (2.10) now equivalently becomes:

$$
\begin{aligned}
&\text{Given } f \in \left[H^1\left(\Omega_h;\mathbb{R}^d\right)\right]'.\\
&\text{Seek } (\boldsymbol{\sigma}, u, \hat{\boldsymbol{\sigma}}, \hat{u}) \in L^2\left(\Omega_h;\mathbb{R}^{d,d}\right) \times L^2\left(\Omega_h;\mathbb{R}^d\right) \times H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)\\
&\text{such that, } \forall (\boldsymbol{\tau}, v) \in H\left(\operatorname{div}, \Omega_h;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega_h;\mathbb{R}^d\right),\\
&\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} \boldsymbol{\sigma} : \left(\mathcal{A}\boldsymbol{\tau} + \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)\right) + u \cdot \left(\frac{1}{2}\nabla \cdot \left(\boldsymbol{\tau} + \boldsymbol{\tau}^T\right) - \nabla \cdot (\boldsymbol{\beta} v) + \mu v\right)\, d\mathbf{x}\\
&+ \sum_{j=1}^{N^{\mathrm{el}}} \frac{1}{2} \int_{\partial K_j} \boldsymbol{\beta} \cdot \mathbf{n}\hat{u} \cdot [\![v]\!]\, ds - \left\langle \frac{1}{2}[\![\boldsymbol{\tau} + \boldsymbol{\tau}^T]\!], \hat{u} \right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}\\
&- \left\langle \hat{\boldsymbol{\sigma}}, [\![v]\!] \right\rangle_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d) \times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)} = \langle f, v \rangle_{[H^1(\Omega_h;\mathbb{R}^d)]' \times H^1(\Omega_h;\mathbb{R}^d)}.
\end{aligned}
\tag{5.7}
$$

On the other hand, if $\boldsymbol{\beta} = \mathbf{0}$, we can identify $q \in H\left(\operatorname{div}, \Omega;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega;\mathbb{R}^d\right)/Q(\Omega)$ with $(\hat{\boldsymbol{\sigma}}, \hat{u}) \in H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)$, and we can use either $[\![\cdot]\!]_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}$ or $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}$ as norm in $H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)$. In this case, the abstract DPG formulation (2.10) now equivalently becomes:

$$
\begin{aligned}
&\text{Given } f \in \left[H^1\left(\Omega_h;\mathbb{R}^d\right)\right]'.\\
&\text{Seek } (\boldsymbol{\sigma}, u, \hat{\boldsymbol{\sigma}}, \hat{u}) \in L^2\left(\Omega_h;\mathbb{R}^{d,d}\right) \times L^2\left(\Omega_h;\mathbb{R}^d\right) \times H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)\\
&\text{such that, } \forall (\boldsymbol{\tau}, v) \in H\left(\operatorname{div}, \Omega_h;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega_h;\mathbb{R}^d\right),\\
&\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} \boldsymbol{\sigma} : \left(\mathcal{A}\boldsymbol{\tau} + \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)\right) + u \cdot \left(\frac{1}{2}\nabla \cdot \left(\boldsymbol{\tau} + \boldsymbol{\tau}^T\right) + \mu v\right)\, d\mathbf{x}\\
&- \left\langle \frac{1}{2}[\![\boldsymbol{\tau} + \boldsymbol{\tau}^T]\!], \hat{u} \right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}\\
&- \left\langle \hat{\boldsymbol{\sigma}}, [\![v]\!] \right\rangle_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d) \times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)} = \langle f, v \rangle_{[H^1(\Omega_h;\mathbb{R}^d)]' \times H^1(\Omega_h;\mathbb{R}^d)}.
\end{aligned}
\tag{5.8}
$$

Consequently, results in Section 2 hold, and in particular, the well-posedness of DPGopt, DPGlopt, and DPGqopt is readily available for both (5.7) and (5.8). The DPGlopt for linear elasticity equations ($\boldsymbol{\beta} = \mathbf{0}$ and $\mu = 0$) is related to the DPG method analyzed in [5], but here in this paper our well-posedness proof is different and comes directly from Section 2. The DPGqopt seems to be new in the context of linear elasticity. A linearized version of the compressible Navier-Stokes equations considered in [20] is corresponding to

$$
\mathcal{A} = I_{d^2} - \frac{1}{d + \lambda}\mathcal{Z},
$$

where $\lambda > 0$ is the compressibility factor, and $\mathcal{Z}_{[ij][kl]} = \delta_{ij}\delta_{kl}$. Compared to the existing DPG method for one dimensional Navier-Stokes equation in [7], our three DPG methods seem to be the first efforts in developing DPG approaches with guaranteed well-posedness to a multi-dimensional linearized version of the Navier-Stokes equations.

5.4. Time domain acoustic equations. In this section, we apply our abstract framework devised in Section 2 to time-domain acoustic equations. Alternatively, one can consider frequency-domain acoustic equations leading to Helmholtz equations for which a DPG method has been proposed and analyzed in [14]. The acoustic equations in the pressure-velocity in time domain have the following form

$$\begin{aligned} \rho c^2 \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0, T_f), \\ \rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0, T_f), \\ p(\mathbf{x}, 0) &= p_0(\mathbf{x}) && \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= \lambda p && \text{in } \partial\Omega \times (0, T_f), \end{aligned}$$

where ρ is the density, c the speed of sound, p the pressure, and $\mathbf{u}$ the velocity vector. There are several approaches to deal with time dependent problems. For example, one can use our DPG framework simultaneously for both space and time to arrive at a space-time DPG formulation (see Chan *et al.* [8] for a space-time DPG formulation of one dimensional convection, convection-diffusion, and Burger's equations). Here, we explore a simple approach to cast the time-dependent acoustic equations into a Friedrichs' system discussed in Section 3. To this end, we first assume that both $\rho \in L^\infty(\Omega)$ and $c \in L^\infty$ are positive and uniformly bounded away from zero. Next, we discretize the time derivative using the backward Euler method to obtain

$$\begin{aligned} \frac{\rho c^2}{\Delta t} p^n + \nabla \cdot \mathbf{u}^n &= \frac{\rho c^2}{\Delta t} p^{n-1} && \text{in } \Omega, \\ \frac{\rho}{\Delta t} \mathbf{u}^n + \nabla p^n &= \mathbf{f} + \frac{\rho}{\Delta t} \mathbf{u}^{n-1} && \text{in } \Omega, \\ p^0(\mathbf{x}, 0) &= p_0(\mathbf{x}) && \text{in } \Omega, \\ \mathbf{u}^0(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega, \\ \mathbf{u}^n \cdot \mathbf{n} &= \lambda p^n && \text{in } \partial\Omega, \end{aligned}$$

where $t_n = n \times \Delta t$. Note that our approach is also valid for other time stepping schemes, but the backward Euler is chosen for simplicity in the exposition. By a straightforward renaming of variables, the acoustic equations at each time step have the following form

$$\begin{aligned} \varepsilon \boldsymbol{\sigma} + \nabla u &= \mathbf{f} && \text{in } \Omega, \\ \mu u + \nabla \cdot \boldsymbol{\sigma} &= g && \text{in } \Omega, \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \lambda u && \text{on } \partial\Omega, \end{aligned}$$

which is clearly a Friedrichs' system discussed in Section 3 with $m = d + 1$ and the corresponding matrices

$$C = \begin{bmatrix} \varepsilon & \mathbf{0} \\ \mathbf{0} & \mu \end{bmatrix}, \quad A^k = \begin{bmatrix} \mathbf{0} & \mathbf{e}^k \\ (\mathbf{e}^k)^T & 0 \end{bmatrix}.$$

Similar to Section 5.2, the graph space is

$$W = H(\text{div}, \Omega) \times H^1(\Omega).$$

The following results, parallel to Lemma 5.3, are easily to inspect.

LEMMA 5.9.

i) The trace operator

$$\gamma : H\left(div, \Omega\right) \times H^1\left(\Omega\right) \ni \left(\boldsymbol{\sigma}, u\right) \mapsto \left(\boldsymbol{\sigma}\cdot\mathbf{n}, u\right) \in H^{-\frac{1}{2}}\left(\partial\Omega\right) \times H^{\frac{1}{2}}\left(\partial\Omega\right)$$

is a continuous surjection satisfying

$$\left\langle B\left(\boldsymbol{\sigma}, u\right), \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = \left\langle \boldsymbol{\sigma}\cdot\mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + \left\langle \boldsymbol{\tau}\cdot\mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}.$$

ii) Define

$$\left\langle M\left(\boldsymbol{\sigma}, u\right), \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = \left\langle \boldsymbol{\tau}\cdot\mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} - \left\langle \boldsymbol{\sigma}\cdot\mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + 2\int_{\partial\Omega} \lambda u v\, ds,$$

then M satisfies (2.5a) and (2.5b). Furthermore,

$$V = \left\{\left(\boldsymbol{\sigma}, u\right) \in H\left(div, \Omega\right) \times H^1\left(\Omega\right) : \boldsymbol{\sigma}\cdot\mathbf{n} = \lambda u \text{ on } \partial\Omega\right\},$$
$$V^* = \left\{\left(\boldsymbol{\sigma}, u\right) \in H\left(div, \Omega\right) \times H^1\left(\Omega\right) : \boldsymbol{\sigma}\cdot\mathbf{n} = -\lambda u \text{ on } \partial\Omega\right\}.$$

For any $q = \left(q^{\boldsymbol{\sigma}}, q^u\right) \in W\left(\Omega\right)$, Lemma 5.9 suggests a natural continuous extension of $\left(M - B\right) q$ from $W'\left(\Omega\right)$ to $W'\left(\Omega_h\right)$ as

$$\left\langle \left(M - B\right) q, \left(\boldsymbol{\tau}, v\right)\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} = -2\left\langle q^{\boldsymbol{\sigma}}\cdot\mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega_h)\times H^{\frac{1}{2}}(\partial\Omega_h)} + 2\int_{\partial\Omega_h} \lambda q^u v\, ds,$$

from which the compatibility condition (2.11) is trivially satisfied.

Next, we study the quotient space $\tilde{W}\left(\Omega\right) = H\left(\text{div}\right) \times H^1/Q\left(\Omega\right)$. Here is a result parallel to Theorem 5.2.

THEOREM 5.10.

i) The subspace Q is given by

$$Q = \left\{q \in H\left(div, \Omega\right) \times H^1\left(\Omega\right) : q^{\boldsymbol{\sigma}}\cdot\mathbf{n} = 0 \text{ on } \Gamma_h^0 \text{ and } q^u = 0 \text{ on } \Gamma_h\right\}.$$

Furthermore, $H\left(div\right) \times H^1/Q\left(\Omega\right)$ is isomorphic to $H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$. In particular, the trace of a function in the quotient space $H\left(div\right) \times H^1/Q\left(\Omega\right)$ is independent of its representations.

ii) For each $\left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \in H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$, define a new norm

$$\left\| \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \right\|_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}(\Gamma_h)} = \left\|[q]\right\|_{H(div)\times H^1/Q(\Omega)},$$

where $[q] \in H\left(div\right) \times H^1/Q\left(\Omega\right)$ such that there exists a representation q satisfying $\gamma q = \left(q^{\boldsymbol{\sigma}}\cdot\mathbf{n}, q^u\right) = \left(\hat{\boldsymbol{\sigma}}, \hat{u}\right)$ on $\Gamma_h^0 \times \Gamma_h$. Then, $\left\|\cdot\right\|_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}(\Gamma_h)}$ is equivalent to $\left\|\cdot\right\|_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}(\Gamma_h)}$, and hence generating the same topology in $H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$. In particular, $H\left(div\right) \times H^1/Q\left(\Omega\right)$ and $H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$ are homeomorphic.

Proof. For this example, one has

$$a\left(q, \left(\boldsymbol{\tau}, v\right)\right) = \left\langle [\![\boldsymbol{\tau}]\!], q^u\right\rangle_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)} + \left\langle q^{\boldsymbol{\sigma}}\cdot\mathbf{n}, [\![v]\!]\right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)} + \int_{\partial\Omega_h} \lambda q^u v\, ds.$$

The rest of the proof is similar to that of Theorem 5.4. ◻

As a direct consequence of Theorem 5.10, we can identify $q \in H\left(\text{div}\right) \times H^1/Q\left(\Omega\right)$ with $\left(\hat{\boldsymbol{\sigma}}, \hat{u}\right) \in H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$, and we can use either $\|\!|\cdot|\!\|_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)}$ or $\left\|\cdot\right\|_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)}$ as norm in $H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right)$. The abstract DPG formulation (2.10) now equivalently becomes, $\forall \left(\boldsymbol{\tau}, v\right) \in H\left(\text{div}, \Omega_h\right) \times H^1\left(\Omega_h\right)$,

$$\begin{aligned}
&\text{Given } \left(f, g\right) \in \left[H\left(\text{div}, \Omega_h\right)\right]' \times \left[H^1\left(\Omega_h\right)\right]'. \\
&\text{Seek } \left(\boldsymbol{\sigma}, u, \hat{\boldsymbol{\sigma}}, \hat{u}\right) \in \left[L^2\left(\Omega_h\right)\right]^d \times L^2\left(\Omega_h\right) \times H^{-\frac{1}{2}}\left(\Gamma_h^0\right) \times H^{\frac{1}{2}}\left(\Gamma_h\right) \text{ such that} \\
&\sum_{j=1}^{N^{\text{el}}} \int_{K_j} \boldsymbol{\sigma} \cdot \left(\varepsilon \boldsymbol{\tau} - \nabla v\right) + u\left(-\nabla \cdot \boldsymbol{\tau} + \mu v\right)\, d\mathbf{x} \\
&+ \left\langle [\![\boldsymbol{\tau}]\!], \hat{u}\right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)} + \left\langle \hat{\boldsymbol{\sigma}}, [\![v]\!]\right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)} + \int_{\partial\Omega_h} \lambda \hat{u} v\, ds \qquad (5.9)\\
&= \left\langle g, v\right\rangle_{\left[H^1\left(\Omega_h\right)\right]'\times\left[H^1\left(\Omega_h\right)\right]} + \left\langle f, \boldsymbol{\tau}\right\rangle_{\left[H\left(\text{div},\Omega_h\right)\right]'\times\left[H\left(\text{div},\Omega_h\right)\right]}.
\end{aligned}$$

Consequently, results in Section 2 hold, and in particular, the well-posedness of DPGopt, DPGlopt, and DPGqopt is readily available for (5.9). Our work is one of the first efforts in developing DPG methods for time-dependent PDEs in general, and the first for time-domain acoustic equations in particular. Since the bilinear form is identical for all time steps, so are the optimal test functions, assuming the trial basis functions are not a function of time. In other words, the optimal test functions, once computed for the first time step, can be used for all subsequent time steps. Another direct consequence is that the stiffness matrix remains the same for all time steps, implying that matrix factorization is only done once if a direct solver is used. Hence, the time-domain DPG methods for acoustic equations proposed in this section are slightly more expensive than the existing DPG methods for steady convection-diffusion problems.

5.5. Maxwell's equations in the elliptic regime. We now apply the abstract theory in Section 2 to a version of the Maxwell's equation considered in [18, 21]. The governing equations in three dimensional space, i.e., $d = 3$, read

$$\begin{aligned}
\mu H + \nabla \times E &= f \quad \text{in } \Omega, \\
\lambda E - \nabla \times H &= g \quad \text{in } \Omega, \\
E \times \mathbf{n} &= 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where $\mu, \lambda \in L^\infty\left(\Omega\right)$ are positive and bounded away from zero. Here, E and H are the electric and the magnetic fields, respectively. Clearly, E, H, f, g are vector-valued function in $\mathbb{R}^3$. One can cast the governing equations into the Friedrichs' framework discussed in Section 3 with $m = 6$ and

$$C = \left[\begin{array}{cc} \mu & \mathbf{0} \\ \mathbf{0} & \lambda \end{array}\right], \quad A^k = \left[\begin{array}{cc} \mathbf{0} & \mathcal{R}^k \\ \left(\mathcal{R}^k\right)^T & \mathbf{0} \end{array}\right],$$

where the components of the Levi-Civita permutation tensor are used to form the matrices $\mathcal{R}^k$, namely, $\mathcal{R}^k_{ij} = \epsilon_{ikj}$, $1 \le i, k, j \le 3$. It is obvious that the graph space is defined as

$$W\left(\Omega\right) = H\left(\text{curl}, \Omega\right) \times H\left(\text{curl}, \Omega\right).$$

Next, we summarize a few results in [18,21].

Lemma 5.11. *The following hold:*

i) The trace operator

$$\gamma : H\left(curl, \Omega\right) \times H\left(curl, \Omega\right) \ni (H,E) \mapsto (H\times \mathbf{n}, E\times \mathbf{n}) \in H^{-\frac{1}{2}}\left(\partial\Omega\right) \times H^{-\frac{1}{2}}\left(\partial\Omega\right)$$

is a continuous surjection satisfying, $\forall\left(h,e\right) \in H^1\left(\Omega\right) \times H^1\left(\Omega\right)$,

$$\begin{aligned}
&\left\langle B\left(H,E\right),\left(h,e\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = \left(\nabla\times E, h\right)_{[L(\Omega)]^3} - \left(E, \nabla\times h\right)_{[L(\Omega)]^3} \\
&+ \left(\nabla\times e, H\right)_{[L(\Omega)]^3} - \left(e, \nabla\times H\right)_{[L(\Omega)]^3} \\
&= \left\langle E\times \mathbf{n}, h\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} - \left\langle H\times \mathbf{n}, e\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

ii) $\forall\left(h,e\right) \in H^1\left(\Omega\right) \times H^1\left(\Omega\right)$, *define*

$$\begin{aligned}
&\left\langle M\left(H,E\right),\left(h,e\right)\right\rangle_{W'(\Omega)\times W(\Omega)} = -\left(\nabla\times E, h\right)_{[L(\Omega)]^3} + \left(E, \nabla\times h\right)_{[L(\Omega)]^3} \\
&+ \left(\nabla\times e, H\right)_{[L(\Omega)]^3} - \left(e, \nabla\times H\right)_{[L(\Omega)]^3} \\
&= -\left\langle E\times \mathbf{n}, h\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} - \left\langle H\times \mathbf{n}, e\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

then M satisfies (2.5a) and (2.5b). Furthermore,

$$V = V^* = \left\{(H,E) \in H\left(curl, \Omega\right) \times H\left(curl, \Omega\right) : \left.(E\times\mathbf{n})\right|_{\partial\Omega} = 0\right\}.$$

Note that the first equality in the definition of M and B is valid for $(h,e) \in H\left(\text{curl}, \Omega\right) \times H\left(\text{curl}, \Omega\right)$, and it is simplified in the second equality when $(h,e) \in H^1\left(\Omega\right) \times H^1\left(\Omega\right)$. Now, for any $q = \left(q^H, q^E\right) \in W\left(\Omega\right)$, we have

$$\left\langle (M-B)\,q, (h,e)\right\rangle_{W'(\Omega)\times W(\Omega)} = -2\left(\nabla\times q^E, h\right)_{[L(\Omega)]^3} + 2\left(q^E, \nabla\times h\right)_{[L(\Omega)]^3},$$

which suggests a natural extension of $(M-B)\,q$ from $W'\left(\Omega\right)$ to $W'\left(\Omega_h\right)$ as

$$\left\langle (M-B)\,q, (h,e)\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} = \sum_{j=1}^{N^{\text{el}}} -2\left(\nabla\times q^E, h\right)_{[L(K_j)]^3} + 2\left(q^E, \nabla\times h\right)_{[L(K_j)]^3}.$$

Thus, the compatibility condition (2.11) is automatically satisfied.

Next, we study the quotient space $\tilde{W}\left(\Omega\right) = H\left(\text{curl}\right) \times H\left(\text{curl}\right)/Q\left(\Omega\right)$. Here is a result parallel to Theorem 5.2.

Theorem 5.12.

i) The subspace Q is given by

$$Q = \left\{q \in H\left(curl, \Omega\right) \times H\left(curl, \Omega\right) : q^H \times \mathbf{n} = 0 \ \text{on}\ \Gamma_h\right\}.$$

Furthermore, $H\left(curl\right) \times H\left(curl\right)/Q\left(\Omega\right)$ is isomorphic to $H^{-\frac{1}{2}}\left(\Gamma_h\right)$. In particular, the trace of a function in the quotient space $H\left(curl\right) \times H\left(curl\right)/Q\left(\Omega\right)$ is independent of its representations.

ii) For each $\hat{H} \in H^{-\frac{1}{2}}\left(\Gamma_h\right)$, define a new norm

$$\Vert \hat{H} \Vert_{H^{-\frac{1}{2}}(\Gamma_h)} = \left\|[q]\right\|_{H(curl)\times H(curl)/Q(\Omega)},$$

where $[q] \in H\left(curl\right) \times H\left(curl\right)/Q\left(\Omega\right)$ such that there exists a representation q satisfying $q^H \times \mathbf{n} = \hat{H}$ on Γ_h. Then, $\Vert \cdot \Vert_{H^{-\frac{1}{2}}(\Gamma_h)}$ is equivalent to $\left\|\cdot\right\|_{H^{-\frac{1}{2}}(\Gamma_h)}$, and hence generating the same topology in $H^{-\frac{1}{2}}\left(\Gamma_h\right)$. In particular, $H\left(curl\right) \times H\left(curl\right)/Q\left(\Omega\right)$ and $H^{-\frac{1}{2}}\left(\Gamma_h\right)$ are homeomorphic.

Proof. For $(h, e) \in H^1(\Omega_h) \times H^1(\Omega_h) \subset H(\text{curl}, \Omega_h) \times H(\text{curl}, \Omega_h)$, the bilinear form $a(q, (h, e))$, by using Lemma 5.11, becomes

$$a(q, (h, e)) = -\sum_{e \in \Gamma_h} \left(q^H \times \mathbf{n}_e, [\![e]\!]\right)_{H^{-\frac{1}{2}}(e) \times H^{\frac{1}{2}}(e)}.$$

Now enforcing $a(q, (h, e)) = 0$ for all $(h, e) \in H^1(\Omega_h) \times H^1(\Omega_h) \subset H(\text{curl}, \Omega_h) \times H(\text{curl}, \Omega_h)$ concludes that $q^H \times \mathbf{n} = 0$ on Γ_h since the trace of $H^1(\Omega_h)$ spans $H^{\frac{1}{2}}(\Gamma_h)$. The rest of the proof is similar to that of Theorem 5.2. ∎

As a direct consequence of Theorem 5.12, the ultra weak formulation (2.10) can be now written equivalently as:

$$\begin{aligned}
&\text{Given } (f, g) \in [H(\text{curl}, \Omega_h) \times H(\text{curl}, \Omega_h)]'. \\
&\text{Seek } (u, q) \in L(\Omega_h) \times H(\text{curl}) \times H(\text{curl})/Q(\Omega) \text{ such that} \\
&\sum_{j=1}^{N^{\text{el}}} \int_{K_j} \left(-u^E \cdot \nabla \times h + u^H \cdot \nabla \times e + q^H \cdot \nabla \times e - e \cdot \nabla \times q^H\right) d\mathbf{x} \\
&= \langle f, h \rangle_{[H(\text{curl},\Omega_h)]' \times H(\text{curl},\Omega_h)} + \langle g, e \rangle_{[H(\text{curl},\Omega_h)]' \times H(\text{curl},\Omega_h)},
\end{aligned} \tag{5.10}$$

for all $(h, e) \in H(\text{curl}, \Omega_h) \times H(\text{curl}, \Omega_h)$. Consequently, results in Section 2 are valid, and in particular, the well-posedness of DPGopt, DPGlopt, and DPGqopt is readily available for (5.10). Our work is the first effort in developing DPG methods for the Maxwell's equations.

Theorem 5.12 suggests that we can identify $q \in H(\text{curl}) \times H(\text{curl})/Q(\Omega)$ with $\hat{H} \in H^{-\frac{1}{2}}(\Gamma_h)$, and we can use either $[\![\cdot]\!]_{H^{-\frac{1}{2}}(\Gamma_h)}$ or $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h)}$ as norm in $H^{-\frac{1}{2}}(\Gamma_h)$. Unlike other problems in previous sections, the new unknown q in (5.10) cannot be substituted by its corresponding $\hat{H}$ since B_{K_j} does not generally have a boundary representation when $(h, e) \in H(\text{curl}, \Omega_h) \times H(\text{curl}, \Omega_h)$. This is, however, possible if (h, e) is restricted in $H^1(\Omega_h) \times H^1(\Omega_h)$. It should be emphasized here that a boundary representation is vital for finite dimensional approximations since one needs to solve for the unknown flux $\hat{H}$ on the skeleton Γ_h instead of q on the whole domain Ω as we now show. Suppose the subspace $\mathcal{V}_r$ introduced in Section 2.4 is a subset of $H^1(\Omega_h)$, then the discrete equation (2.21) equivalently reads

$$\begin{aligned}
&\text{Given } (f, g) \in [H(\text{curl}, \Omega_h) \times H(\text{curl}, \Omega_h)]'. \\
&\text{Seek } \left(u, \hat{H}\right) \in \mathcal{U}_N \subset L(\Omega_h) \times H^{-\frac{1}{2}}(\Gamma_h) \text{ such that, } \forall (h, e) \in \mathcal{V}_r, \\
&\sum_{j=1}^{N^{\text{el}}} \int_{K_j} \left(-u^E \cdot \nabla \times h + u^H \cdot \nabla \times e\right) d\mathbf{x} - \left\langle \hat{H}, [\![e]\!] \right\rangle_{H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h)} \\
&= \langle f, h \rangle_{[H(\text{curl},\Omega_h)]' \times H(\text{curl},\Omega_h)} + \langle g, e \rangle_{[H(\text{curl},\Omega_h)]' \times H(\text{curl},\Omega_h)},
\end{aligned} \tag{5.11}$$

6. Conclusions. We have proposed a unified discontinuous Petrov–Galerkin (DPG) framework for Friedrichs-like systems, which embrace a large class of elliptic, parabolic, and hyperbolic partial differential equations (PDEs). The well-posedness, i.e., existence, uniqueness, and stability, of the DPG solution is established on a single abstract DPG formulation, and three abstract DPG methods corresponding to three different, but equivalent, norms are devised. We have then applied the single DPG framework to several linear(ized) PDEs including, but not limited to, scalar transport, Laplace, diffusion, convection-diffusion, convection-diffusion-reaction, linear(ized) continuum mechanics (e.g., linear(ized) elasticity, a version of the linearized Navier-Stokes equations, and etc), time-domain acoustics, and a version of the Maxwell's equations. The results show that we not only recover several existing DPG methods, but also discover new DPG methods for both PDEs currently considered in the DPG community and new ones. As a direct consequence of the single abstract DPG framework, all the DPG methods have been shown to be trivially well-posed.

Ongoing research is to apply the abstract framework to the linearized Euler and compressible Navier-Stokes equations. On the other hand, since the setting is in real Hilbert spaces, our methodology cannot be directly applied to the Helmholtz equations. One of our future directions is therefore to modify the theory to complex Hilbert spaces.

Acknowledgements. We are thankful to Professor Jean-Luc Guermond for fruitful discussions on the trace operator in Lemma 5.1. We also thank Professors Jayadeep Gopalakrishnan and Ignacio Muga for valuable discussions and comments on the first draft of the paper.

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